Etude mathématique et numérique de modèles issus du domaine biomédical

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Mémoire

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Contents

Introduction générale

Ce mémoire présente les travaux de recherche réalisés depuis la fin de mon doctorat. Ils s'articulent autour de trois thématiques :

- l'analyse mathématique des problèmes d'interaction fluide-structure qui modélisent l'écoulement du sang dans les artères (parmi de nombreux autres phénomènes !),
- l'identification de paramètres pour un modèle simplifié d'écoulement de l'air dans l'arbre respiratoire,
- l'étude mathématique et numérique de modèles d'électrophysiologie cardiaque.

Ces thèmes ont donc pour point commun de considérer des modèles issus du domaine biomédical.

Dans la première partie de ce mémoire, les travaux présentés se situent dans le prolongement de ceux réalisés durant mon doctorat et sont des contributions à l'analyse mathématique des problèmes d'interaction fluide-structure. Le fluide est modélisé par les équations de Navier-Stokes incompressible ou compressible et la structure est rigide ou élastique. Une série de travaux étudie l'existence et l'unicité de solution régulière. On aborde aussi l'étude de la contrôlabilité locale à zéro du problème d'interaction entre un fluide incompressible et une structure rigide : en considérant un contrôle qui agit sur une partie du domaine fluide, on cherche à amener l'ensemble fluide-structure au repos et à amener la structure à une certaine position.

La seconde partie est consacrée aux problèmes inverses venant de la modélisation de l'air dans l'arbre respiratoire. On considère ici les équations de Stokes et l'objectif est d'identifier un cœfficient de Robin (correspondant à une résistance dans l'arbre bronchique) qui intervient dans les conditions au bord sur une partie de la frontière à partir de mesures réalisées sur une autre partie de la frontière. Les résultats reposent sur des inégalités de Carleman locales et donnent une stabilité logarithmique des cœfficients de Robin par rapport aux observations.

Enfin, la troisième partie résume les résultats obtenus dans le domaine d'application de l'électrophysiologie cardiaque. Différentes sujets sont abordés : l'analyse mathématique des problèmes direct et inverse, l'obtention par des simulations numériques 3D d'électrocardiogrammes réalistes, l'identification numérique de paramètres et l'impact du bruit.

General introduction

This report presents the research works achieved since the end of my PhD. They may be organized in three themes:

- the mathematical analysis of fluid-structure interaction problems which model the blood flow in the arteries (among many other phenomena !),
- the parameter identification for a simplified model of the airflow in the respiratory tract,
- the mathematical and numerical study of models in cardiac electrophysiology.

Thus, in all these themes, we consider models coming from the biomedical domain.

In the first part of this report, the presented works extend the ones achieved during my PhD and deal with the mathematical analysis of fluid-structure interaction problems. The fluid is modeled by the incompressible or compressible Navier-Stokes equations and the structure is rigid or elastic. A succession of works study the existence and uniqueness of regular solution. We also tackle the local null controllability of the interaction problem between an incompressible fluid and a rigid structure: thanks to a control which acts on a part of the fluid domain, we drive the fluid-structure system at rest and the structure at its reference configuration.

The second part is dedicated to inverse problems coming from the modeling of the respiratory airflow. We consider the Stokes equations and our objective is to identify a Robin coefficient (corresponding to a resistance in the bronchial tree) which appears in the boundary conditions on a part of the boundary from measurements made on another part of the boundary. These results rely on local Carleman inequalities and give the logarithmic stability of the Robin coefficients with respect to the measurements.

At last, the third part summarizes results obtained in cardiac electrophysiology. Different subjects have been tackled: the mathematical analysis of the direct and inverse problems, 3D numerical simulations and their validation by realistic electrocardiograms, the numerical identification of parameters and the impact of noise.

Chapter 1

Well-posedness and controllability for fluid-structure interaction problems

Fluid-structure interaction problems may describe various phenomena which are represented by different kinds of geometries and mathematical models. Depending on the situation, we may for instance be interested by the evolution of a fluid inside an elastic wall (blood flow in the arteries), the evolution of a rigid structure in a compressible fluid (plane in the sky) or in an incompressible fluid (submarine in the ocean) or the evolution of an elastic structure in a fluid (organism swimming in the water). From a mathematical and numerical point of view, studying these problems raises many difficulties which often require to set up specific techniques.

In the introduction of this chapter, I will try to present in a general way the models I have been interested in and to highlight the difficulties raised by the analysis of these problems. Then, I will present my contributions on the well-posedness of fluid-structure interaction problems in Section 2 and on their controllability in Section 3.

1 Introduction

We commonly use the eulerian formulation to describe the fluid flow. This means that the fluid motion is described by the time evolution of the velocity field when we stay at a fixed point of the space (we do not follow the fluid particles, contrary to the lagrangian formulation).

Let us give the equations which describe the evolution of the fluid density ρ, the fluid velocity u and the fluid pressure p, quantities which are evaluated at time t and at the geometric point x called eulerian variable. First, according to the equation of the mass conservation, we have: for all $t > 0$, for all $x \in \Omega_F(t)$

$$\partial_t \rho(t,x) + \nabla \cdot (\rho u)(t,x) = 0. \tag{1.1}$$

Next, the equation of conservation of momentum in its conservative form is given by: for all $t > 0$, for all $x \in \Omega_F(t)$

$$\partial_t(\rho u)(t,x) + \nabla \cdot (\rho u \otimes u)(t,x) - \nabla \cdot \sigma(u,p)(t,x) = 0$$

where the Cauchy stress tensor $\sigma(u,p)$ is determined by the kinematic properties of the fluid. In the following, we will specify its expression, depending on the problem that will be tackled: we will consider newtonian viscous incompressible or compressible fluids. This equation can be rewritten, by using the mass conservation equation

$$(\rho \partial_t u + \rho(u \cdot \nabla)u)(t,x) - \nabla \cdot \sigma(u,p)(t,x) = 0. \tag{1.2}$$

If the fluid interacts with a structure, equations (1.1) and (1.2) are given on a domain $\Omega_F(t)$ which evolves with respect to time and which depends on the domain occupied by the structure which we denote by $\Omega_S(t)$. Since this domain is unknown, the domain where the fluid equations are set is unknown itself. Dealing with a free boundary problem corresponds to the first difficulty of the fluid-structure interaction studies.

Unlike the fluid, the equations of the solid motion are written in a natural way in the lagrangian formulation. Intuitively, this difference of point of view expresses mechanic behaviors of different nature: in the absence of strength, an elastic structure which has been subjected to a strength comes back to its initial position. It is thus possible to "follow" the deformation of a solid particle which usually stays relatively small, contrary to the case of a fluid particle. This difference of point of view appears in the expression of the coupling conditions between the fluid and the structure.

The coupling conditions are expressed through conditions at the fluid-structure interface. The first one is a dynamic condition which enforces the continuity of the normal component of the stress tensor, according to the law of reciprocal action. The second condition is a kinematic condition on the velocities. If the fluid is assumed to be viscous, the velocity is continuous at the interface which is expressed by the condition:

$$u(t, \chi_F(t, y)) = v_S(t, y), \, \forall \, (t, y) \in (0, T) \times \partial\Omega_S(0) \cap \partial\Omega_F(0). \tag{1.3}$$

Here, $v_S(t, y)$ is the lagrangian velocity of the solid at time t and at the lagrangian point y of the reference configuration. In this equality, we have used the flow function χ_F which allows to go from an eulerian point of view to a lagrangian point of view. It is defined for all $(t, y) \in (0, T) \times \Omega_F(0)$ by

$$\begin{cases} \partial_t \chi_F(t, y) & = u(t, \chi_F(t, y)) \\ \chi_F(0, y) & = y. \end{cases}$$

According to (1.3), at the fluid-structure interface, χ_F matches with the solid flow χ_S defined by:

$$\chi_S(t, y) = y + \int_0^t v_S(t', y) \, dt', \, \forall \, (t, y) \in (0, T) \times \Omega_S(0).$$

A classical method to overcome the fact that the fluid equations are set on an unknown domain which depends on time is to make a change of variables in order to write the fluid equations on a fixed domain. This change of variables can be done thanks to the flow χ_F which describes the real motion of the fluid particles. But, more generally, we can choose any flow χ which coincides with the solid or fluid flow at the interface and which stays invertible at least locally in time in order to bring back the domain $\Omega_F(t)$ to $\Omega_F(0)$. The choice that we have in the definition of the flow χ is the basis of the ALE method which is one of the most used numerical methods for the fluid-structure interaction problems. For a presentation of this method, we refer for instance to [QTV00] and [FFGQ09] (this last reference also explains the derivation of different fluid-structure interaction models from mechanical principles).
In particular, if the structure is rigid, it is much easier to define χ as an extension of the solid flow. By this way, one can easily construct a flow which will be very regular in space (since it is the case for χ_S) and invertible at a fixed time t.

In the works that are presented here, we assume that the structure is rigid or elastic. As we will see, many additional difficulties appear for the mathematical study of the coupled problem when the structure is elastic.

Let us assume, to start with, that the structure is rigid. Then, its motion is described by a translation vector a and a rotation matrix Q. Without any restriction, we can always assume that $a(0) = 0$ and $Q(0) = \mathrm{Id}$. Let us define the solid flow, which associates to a particle initially located at a point y its position at an arbitrary time t:

$$\chi_S(t, y) = a(t) + Q(t)y, \; \forall\, y \in \Omega_S(0). \tag{1.4}$$

The time t being fixed, we see that we can easily invert the flow and we will thus be able to change from point of view without any difficulty. For all $t > 0$, we have

$$\chi_S^{-1}(t, x) = Q(t)^T(x - a(t)), \; \forall\, x \in \Omega_S(t)$$

where

$$\Omega_S(t) = \chi_S(t, \Omega_S(0)) = a(t) + Q(t)\Omega_S(0).$$

Here and in what follows, by abuse of notation, we denote by χ_S^{-1} the inverse of the application $\chi_S(t, \cdot)$ at a fixed time t.

To give the equations for the structure, we first introduce ω the rotation vector associated to Q defined by

$$\dot{Q}(t)Q(t)^T y = \omega(t) \wedge y \text{ for all } y \in \mathbb{R}^3. \tag{1.5}$$

The motion of the structure is described by the law of reciprocal actions:

$$m\ddot{a} = \int_{\partial\Omega_S(t)} \sigma(u, p)\, n \, d\gamma \tag{1.6}$$

$$J\dot{\omega} = (J\omega) \wedge \omega + \int_{\partial\Omega_S(t)} (x - a) \wedge \big(\sigma(u, p)\, n\big)\, d\gamma. \tag{1.7}$$

In these formula, the vector n is the outward unit normal vector to $\Omega_S(t)$, the mass $m > 0$ is defined by

$$m := \int_{\Omega_S(0)} \rho_{0,S}(y)\, dy$$

and for all $t > 0$ the moment of inertia $J(t) \in \mathcal{M}_3(\mathbb{R})$ is defined by

$$J(t)_{ij} := \int_{\Omega_S(0)} \rho_{0,S}(y)\big(e_i \wedge Q(t)y\big) \cdot \big(e_j \wedge Q(t)y\big)\, dy \tag{1.8}$$

where $\rho_{0,S}$ is the initial density of the solid.

By this way, a and ω are solution of ordinary differential equations which involve the unknowns of the fluid motion. On the other hand, the Dirichlet boundary conditions (1.3) satisfied by u involve v_S the lagrangian velocity of the structure given by:

$$v_S(t, y) = \dot{a}(t) + \omega(t) \wedge (Q(t)y).$$

Even if v_S is unknown, it has a very specific writing. This point will be very important for the controllability results presented in Section 3: the Carleman inequalities that we will show would not be satisfied for the Navier-Stokes equations with arbitrary boundary conditions of Dirichlet type. Moreover, the solid velocity is very regular in space. Contrary to the case of an elastic structure, it is possible to prove the existence of strong solutions for the coupled problem in the same functional framework as for a fluid alone. For instance, in the result obtained in [BG09] which is described in Subsection 2.1, we consider the same kind of functional spaces as in the paper [MN82] which studies a compressible fluid alone.

When the structure is elastic, we have to face additional difficulties that we quickly present here. They are presented more in details in paragraph 2.2.1.

If we are looking for a weak solution, the standard a priori energy estimate for the elasticity equations implies that the elastic deformation belongs to $L^\infty(H^1)$. This immediately causes several problems: if the sets $\Omega_S(t)$ and $\Omega_F(t)$ only have a regularity in H^1, we are far from being able to make a study in the classical framework for the Navier-Stokes equations. It is also not possible to change from point of view between the eulerian and lagrangian descriptions.

By this way, we have to look for a framework which allows to get elastic deformations more regular than the regularity given by the energy spaces. To do so, a first method consists of regularizing the elasticity equations as in [Bou07], [BST12], [CDEG05], [DEGLT01], [BdV04] and [MC13] when the fluid is incompressible and in [Bou05] when the fluid is compressible.

A second method consists of considering the original problem without any regularization in the elasticity equation and to prove the existence of regular solution. The coupling between equations of hyperbolic and parabolic type induces a mismatch in the regularity properties of the fluid equation and of the solid equation and this makes the study of the existence of strong solutions tricky. Among the works which prove this kind of result for the interaction with an incompressible fluid, we can quote [CS05], [CS06], [Gra08], [KT12a] and [RV14] and for the interaction with a compressible fluid, we can quote [BG10] and [KT12b].

In this chapter, the presentation of the problem is limited to the context of the contributed papers. I refer to the recent book chapter [GLMN14] for a more general presentation on fluid-structure interaction problems (with a special focus on 2D-1D interaction problem).

The rest of this chapter is organized in two parts. In Section 2, I present works which focus on the well-posedness of fluid-structure interaction problems and more precisely on their regularity. In Section 3, I am interested by the controllability of interaction problems between an incompressible fluid and a rigid structure.

2 Well-posedness of the coupled problem - [BG09], [BG10], [BST12]

The works which I will present here extend the works made during my PhD supervised by Jean-Pierre Puel. In the course of my doctorate, I have worked on the existence of weak solutions for interaction problems between an elastic structure and an incompressible or compressible fluid. The papers presented in this part have in common to study the existence and uniqueness of strong solution. Depending on the cases, we will consider incompressible or compressible fluids and rigid or elastic structures.

In all this part, we consider that the whole set composed by the fluid and the structure lies in a fixed and bounded domain Ω in dimension 3. At a given time t, we denote by $\Omega_S(t) \subset \Omega$ the domain occupied by the structure and by $\Omega_F(t) = \Omega \setminus \overline{\Omega_S(t)}$ the domain occupied by the fluid. We assume that Ω and $\Omega_S(0)$ are regular enough and satisfy

$$d(\partial\Omega, \Omega_S(0)) > 0 \tag{1.9}$$

where d corresponds to the distance between 2 sets.

2.1 Interaction between a compressible fluid and a rigid structure [BG09]

2.1.1 Presentation of the problem

We are first going to present the equations considered in the paper [BG09] which has been made in collaboration with Sergio Guerrero.

The equations of the fluid are given by (1.1) and (1.2) where $\sigma(u,p)$ is given by

$$\sigma(u,p) = 2\mu\epsilon(u) + \mu'(\nabla \cdot u)\mathrm{Id} - p\mathrm{Id} \tag{1.10}$$

with $\epsilon(u) = \frac{1}{2}(\nabla u + \nabla u^T)$. The viscosity coefficients μ and μ' are real constants which satisfy :

$$\mu > 0,\ 2\mu + 3\mu' \geqslant 0. \tag{1.11}$$

System (1.1)-(1.2) is complemented by a state equation for the pressure law. We assume that the fluid is barotropic which implies that the pressure only depends on the density:

$$p = P(\rho) \tag{1.12}$$

where P is a function in C^∞ on $\mathbb{R}_+^*$ which satisfies

$$P(\rho) > 0 \text{ and } P'(\rho) > 0,\ \forall\, \rho > 0. \tag{1.13}$$

In particular, these hypotheses include the case of an isentropic flow of a perfect gas where

$$p = C_0\rho^\gamma \tag{1.14}$$

with $\gamma > 0$ the adiabatic constant of the gas.

For a fluid alone on a fixed domain, many works deal with the study of the well-posedness for the compressible Navier-Stokes equations. The existence of weak solutions defined globally in time for these equations was still a open problem at the end of the 80's. The first results which give the existence of global weak solution without any restriction on the data date back to the works by P.L. Lions which are gathered in the book [Lio96]. The existence of weak solution is proved for an adiabatic constant $\gamma \geqslant \frac{9}{5}$ in dimension 3. The proof of this result employs new tools like the notions of renormalized solutions for the continuity equation introduced by [DL89] and of effective viscous flux introduced by [Lio93]. These notions which have been reused in most of the later works play a key role in the proof. Since [Lio96], many studies have tried to get similar results with weaker hypotheses on the adiabatic constant, in order to be closer to physical values (which correspond to $1 \leqslant \gamma \leqslant \frac{5}{3}$). In particular, we quote the works by Feireisl [Fei01], [Fei04] which use the notion of oscillations defect measure for the density and which give existence results for a constant $\gamma > \frac{3}{2}$ in dimension 3.

Concerning the existence of strong solution for a compressible fluid alone, results are very incomplete. In particular, the results which can be found in the literature give the local existence of very smooth solutions. In [MN82], Matsumura and Nishida show, for small data, the global existence and uniqueness of strong solution in dimension 3 in the following sets

$$\rho \in C(0,T;H^3(\Omega)) \cap C^1(0,T;H^2(\Omega)),\ u \in C(0,T;H^3(\Omega)) \cap C^1(0,T;H^1(\Omega)). \tag{1.15}$$

Their method lies on energy estimates of high order. No transitional result has been proven at the level of the regularity of the solutions, unlike the incompressible Navier-Stokes equations where one can prove that the velocity satisfies regularity properties of parabolic type ($u \in H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^2(\Omega))$). The existence of strong solutions for a compressible fluid alone is much more complex than in the incompressible case. This is due to the coupling between an equation of transport type and an equation of parabolic type which appears through strong nonlinearities.

Regarding the studies on the compressible Navier-Stokes equations in a free boundary domain, we quote the paper by Zadrzynska [Zad04] which gives a general presentation of the problem and makes a note of the obtained results.

Concerning the rigid structure, at time t, its motion is described by the flow (1.4). If we denote by ω the angular velocity vector defined by (1.5), then a and ω satisfy the laws of conservation (1.6) and (1.7).

The fluid equations are completed by no-slip Dirichlet boundary conditions: we have, for all $t > 0$

$$\begin{cases} u(t,x) &= 0, \ \forall\, x \in \partial\Omega, \\ u(t,x) &= u_S(t,x), \ \forall\, x \in \partial\Omega_S(t) \end{cases} \tag{1.16}$$

where u_S is the eulerian velocity of the structure given by: $\forall\, t > 0$, $\forall\, x \in \Omega_S(t)$,

$$u_S(t,x) = \dot{a}(t) + \omega(t) \wedge (x - a(t)). \tag{1.17}$$

Finally, the system is completed by initial conditions

$$u(0,\cdot) = u_0 \text{ in } \Omega_F(0),\ \rho(0,\cdot) = \rho_0 \text{ in } \Omega_F(0),\ \dot{a}(0) = a_0,\ \omega(0) = \omega_0. \tag{1.18}$$

We thus consider the system formed by the equations (1.1), (1.2), (1.6), (1.7), (1.16) and (1.18), completed by the laws (1.10) and (1.12).

The existence of weak solutions for this system has been studied in dimension 2 or 3 by Desjardins-Esteban [DE00] and by Feireisl [Fei03a] for a state law like (1.14).

The paper [DE00] shows the existence of weak solutions as long as there is no collision between two structures or between a structure and the external boundary. The adiabatic constant satisfies the condition $\gamma \geqslant 2$. The strong convergence of the density is obtained thanks to techniques introduced in the work by di Perna-Lions [DL89] and in the book by Lions [Lio96].

The result obtained by [Fei03a] holds for $\gamma > 3/2$. The solution is defined globally in time, independently from the potential collisions. The technique to take into account the structure (which is used in San Martin-Starovoitov-Tucsnak [SMST02]) consists in approximating the structures by high viscosity fluids. Then, the scheme of the proof is similar to the one in Feireisl-Novotný-Petzeltová [FNP01] for a compressible fluid alone: an artificial viscosity term is added to the continuity equation (1.1) and an artificial pressure term allows to regularize the solution of the fluid problem.

2.1.2 Description of the result obtained in [BG09]

I will now present the result contained in the paper [BG09]. This result gives the existence and uniqueness of strong solution globally defined under the assumption that the initial conditions are small. More precisely, the solution is defined up to a time T which can be arbitrary large as long as there is no collision i.e. T has to satisfy the following property: there exists $\alpha > 0$ such that

$$\forall\, t \in [0,T],\ d(\Omega_S(t), \partial\Omega) \geqslant \alpha > 0. \tag{1.19}$$

Regarding the regularity of the solutions, we have considered spaces similar to the ones of the paper [MN82] which considers a compressible fluid alone and proves (1.15) for the unknowns ρ and u. However, we have to specify the definition of the spaces to which the solution will belong since, in our case, the spatial domain is not a fixed domain but a domain which depends on time.

Let us consider a function $\chi : [0,T] \times \overline{\Omega_F(0)} \mapsto \mathbb{R}^3$ which belongs to $H^3([0,T]; C^\infty(\overline{\Omega_F(0)}))$ such that, for all $t \in [0,T]$, $\chi(t,\cdot)$ is a C^∞-diffeomorphism from $\overline{\Omega_F(0)}$ to $\overline{\Omega_F(t)}$. For all function $u(t,\cdot) : \Omega_F(t) \mapsto \mathbb{R}^3$, we denote by $U(t,\cdot) : \Omega_F(0) \mapsto \mathbb{R}^3$ the function defined by

$$U(t,y) = u(t,\chi(t,y)).$$

Assume that the final time T is given, we then define the following spaces:

$$H^r_T(H^p) = \{u/U \in H^r(0,T; H^p(\Omega_F(0)))\}, \; C^r_T(H^p) = \{u/U \in C^r(0,T; H^p(\Omega_F(0)))\} \tag{1.20}$$

for all $r = 0,1,2$ and for all $p \in \mathbb{N}$. To show our regularity result, we have to enforce compatibility conditions on the initial conditions. Roughly speaking, they express the fact that the boundary conditions satisfied by u and $\partial_t u$ have to hold at the initial time due to the regularity of the solutions. Even if their writing is technical, these conditions (which are not explicitly given here) are natural and mean that the initial conditions may be seen as the result of the evolution of the system considered from an earlier time. By this way, these initial conditions are not arbitrary.

Under these compatibility conditions, we prove the following result:

Theorem 1.1 *We assume that the hypotheses (1.9), (1.11) and (1.13) are satisfied and that* $\rho_0 \in H^3(\Omega_F(0))$ *and* $u_0 \in H^3(\Omega_F(0))$. *We denote by* $\overline{\rho}$ *the mean of* ρ_0 *in* $\Omega_F(0)$. *Then, there exists* $\delta > 0$ *such that, if*

$$\|\rho_0 - \overline{\rho}\|_{H^3(\Omega_F(0))} + \|u_0\|_{H^3(\Omega_F(0))} + |a_0| + |\omega_0| < \delta, \tag{1.21}$$

the system of equations (1.1), (1.2), (1.6), (1.7), (1.16) and (1.18) has a unique solution (ρ,u,a,ω) *defined on* $(0,T)$ *for all* T *such that the condition (1.19) is satisfied. Moreover, this solution belongs to the following spaces*

$$\begin{aligned} \rho &\in L^2_T(H^3) \cap C^0_T(H^3) \cap H^1_T(H^2) \cap C^1_T(H^2) \cap H^2_T(L^2), \\ u &\in L^2_T(H^4) \cap C^0_T(H^3) \cap C^1_T(H^1) \cap H^2_T(L^2), \\ \dot{a} &\in H^2(0,T) \cap C^1([0,T]), \; \omega \in H^2(0,T) \cap C^1([0,T]). \end{aligned} \tag{1.22}$$

and there exists a positive constant C_1 *independent from* T *such that*

$$N_T(\rho - \overline{\rho}, u, a, \omega) \leqslant C_1(\|\rho_0 - \overline{\rho}\|_{H^3(\Omega_F(0))} + \|u_0\|_{H^3(\Omega_F(0))} + |a_0| + |\omega_0|) \tag{1.23}$$

where N_T *corresponds to the norm associated to the spaces given by (1.22).*

Remark 1.2 *In Theorem 1.1, we obtain the existence of a regular solution defined as long as condition (1.19) is satisfied. We can be more precise by saying that we will be in one of the two following cases: either*

$$T = +\infty$$

or

$$\lim_{t \to T} d(\partial\Omega, \overline{\Omega}_S(t)) = 0.$$

Remark that, according to estimate (1.23), the condition (1.19) will be satisfied on an interval $(0, T_{min})$ *where* $T_{min} > 0$ *only depends on* C_1, δ *and* $d(\partial\Omega, \overline{\Omega}_S(0))$. *Thus, we get at least the existence of a solution defined locally in time.*

Remark 1.3 *In the statement of Theorem 1.1, the spaces to which the solution belongs are sometimes redundant (for instance, since ω belongs to $H^2(0,T)$, it implies that it belongs to $C^1([0,T])$). But this information is no more redundant in the inequality (1.23). For instance, if we consider the embedding $H^1(0,T) \subset C^0([0,T])$, we have the associated inequality*

$$\|u\|_{C^0([0,T])} \leqslant C(T)\|u\|_{H^1(0,T)}$$

where the constant C explodes when T tends to 0. In the first step of the proof, the existence of solution up to a small time is shown thanks to a fixed point argument and it is very important not to use such kind of inequality. This difficulty is overcome by taking in $H^1(0,T)$ the following norm

$$\|u\|_{H^1(0,T)} + \|u\|_{C^0([0,T])}.$$

2.1.3 Scheme of the proof

In a general way, fluid-structure interaction problems are strong nonlinear problems. To obtain the existence of solution, it is usually necessary to linearize the problem and to prove the existence of a fixed point. This classically leads to the existence of a solution defined locally. To get a global solution defined up to an arbitrary time, it is necessary to iterate the construction of a local solution. In our case, the result of local existence holds for small initial data, thus, to be able to iterate, it is necessary to have uniform estimates which insure that the solution will stay small during the existence time and it is necessary to construct the local solution on an interval of fixed size. Thus, the proof of Theorem 1.1 relies on the proof of two intermediate results: a result of local existence (and uniqueness) of the solution and a global estimate of the solution.

The first intermediate result gives the uniqueness and local existence of solution for small initial data and an estimate of the solution under the form

$$N_\tau(\rho - \overline{\rho}, u, a, \omega) \leqslant C_1(\|\rho_0 - \overline{\rho}\|_{H^3(\Omega_F(0))} + \|u_0\|_{H^3(\Omega_F(0))} + |a_0| + |\omega_0|)$$

where τ corresponds to the existence time of the local solution. To prove this result, the first step consists of doing a change of variables in the equations of the fluid in order to set them in a fixed domain. We thus define the flow χ which extends on the whole domain Ω the flow (1.4) defined on $\Omega_S(0)$. This flow has to keep the external boundary $\partial\Omega$ fixed. More precisely, a flow χ is constructed in $H^3(0,T;C^\infty(\overline{\Omega}))$ from the data of $a \in H^3(0,T)$ and $\omega \in H^2(0,T)$ and satisfies, for all $t \in (0,T)$

$$\chi(t,y) = a(t) + Q(t)y, \quad \forall\, y \in \Omega_S(0) \cup \{y \in \Omega, d(y, \partial\Omega_S(0)) < \alpha/4\},$$

$$\chi(t,y) = y, \quad \forall\, y \in \Omega \text{ such that } d(y,\partial\Omega) < \alpha/4$$

where α appears in (1.19).

From this flow, we can define the new unknowns:

$$\tilde{u}(t,y) = u(t,\chi(t,y)), \tilde{\rho}(t,y) = \rho(t,\chi(t,y)) - \overline{\rho}, \forall\,(t,y) \in (0,T) \times \Omega_F(0).$$

The system of equations (1.1), (1.2), (1.6), (1.7), (1.16) and (1.18) can then be rewritten after a change of variables:

$$\begin{cases}
\partial_t\tilde{\rho} + ((\nabla\chi)^{-1}(\tilde{u} - \partial_t\chi)) \cdot \nabla\tilde{\rho} + \overline{\rho}\,\nabla \cdot \tilde{u} = g_0(\tilde{\rho}, \tilde{u}, a, \omega) & \text{in } (0,T) \times \Omega_F(0),\\
\partial_t\tilde{u} - \nabla \cdot \sigma(\tilde{u}, P(\tilde{\rho})) = g_1(\tilde{\rho}, \tilde{u}, a, \omega) & \text{in } (0,T) \times \Omega_F(0),\\
m\ddot{a} = \displaystyle\int_{\partial\Omega_S(0)} \sigma(\tilde{u}, P(\tilde{\rho}))n\,d\gamma + g_2(\tilde{\rho}, \tilde{u}, a, \omega) & \text{in } (0,T),\\
J\dot{\omega} = \displaystyle\int_{\partial\Omega_S(0)} (Qy) \wedge (\sigma(\tilde{u}, P(\tilde{\rho}))n)\,d\gamma + g_3(\tilde{\rho}, \tilde{u}, a, \omega) & \text{in } (0,T),\\
\tilde{u} = 0 & \text{on } (0,T) \times \partial\Omega,\\
\tilde{u} = \dot{a} + \omega \wedge (Qy) & \text{on } (0,T) \times \partial\Omega_S(0),\\
\tilde{\rho}(0,\cdot) = \rho_0 - \overline{\rho},\ \ \tilde{u}(0,\cdot) = u_0 & \text{in } \Omega_F(0),\\
a(0) = 0,\ \ \dot{a}(0) = a_0,\ \ \omega(0) = \omega_0,
\end{cases}$$

where the functions g_0, g_1, g_2 and g_3, which are not given explicitly here, are functions at least quadratic of the quantities $\tilde{\rho}$, $\tilde{u}$, $(\nabla\chi)^{-1} - \mathrm{Id}$, $Q - \mathrm{Id}$ and $\partial_t\chi$. This property is important because it allows to see these functions as remainder terms since the data are small. To prove that the system admits a unique local solution on an interval $(0,\tau)$, we use a fixed-point argument. Let X_1 and X_2 be defined by:

$$X_1 = C^0(0,\tau; H^3(\Omega_F(0))) \cap C^1(0,\tau; H^2(\Omega_F(0))) \cap H^2(0,\tau; L^2(\Omega_F(0)))$$

and

$$X_2 = L^2(0,\tau; H^4(\Omega_F(0))) \cap H^2(0,\tau; L^2(\Omega_F(0))).$$

Let $\hat{\rho} \in X_1$, $\hat{u} \in X_2$, $\hat{a} \in H^3(0,\tau)$ and $\hat{\omega} \in H^2(0,\tau)$ be given. We then define the following linearized system:

$$\begin{cases}
\partial_t\tilde{\rho} + ((\nabla\hat{\chi})^{-1}(\hat{u} - \partial_t\hat{\chi})) \cdot \nabla\tilde{\rho} = g_0(\hat{\rho}, \hat{u}, \hat{a}, \hat{\omega}) - \overline{\rho}\,\nabla \cdot \tilde{u} & \text{in } (0,T) \times \Omega_F(0),\\
\partial_t\tilde{u} - \nabla \cdot (2\mu\epsilon(\tilde{u}) + \mu'(\nabla \cdot \tilde{u})\mathrm{Id}) = g_1(\hat{\rho}, \hat{u}, \hat{a}, \hat{\omega}) - p^0\nabla\tilde{\rho} & \text{in } (0,T) \times \Omega_F(0),\\
m\ddot{a} = \displaystyle\int_{\partial\Omega_S(0)} \sigma(\tilde{u}, P(\tilde{\rho}))n\,d\gamma + g_2(\hat{\rho}, \hat{u}, \hat{a}, \hat{\omega}) & \text{in } (0,T),\\
\hat{J}\dot{\omega} = \displaystyle\int_{\partial\Omega_S(0)} (\hat{Q}y) \wedge (\sigma(\tilde{u}, P(\tilde{\rho}))n)\,d\gamma + g_3(\hat{\rho}, \hat{u}, \hat{a}, \hat{\omega}) & \text{in } (0,T),\\
\tilde{u} = 0 & \text{on } (0,T) \times \partial\Omega,\\
\tilde{u} = \dot{a} + \omega \wedge (\hat{Q}y) & \text{on } (0,T) \times \partial\Omega_S(0),\\
\tilde{\rho}(0,\cdot) = \rho_0 - \overline{\rho},\ \ \tilde{u}(0,\cdot) = u_0 & \text{in } \Omega_F(0),\\
a(0) = 0,\ \ \dot{a}(0) = a_0,\ \ \omega(0) = \omega_0
\end{cases} \tag{1.24}$$

where p^0 is given by:

$$p^0 = \frac{P'(\overline{\rho})}{\overline{\rho}}.$$

Notice that the first two equations are still coupled through linear terms (the last terms in the right-hand sides) which can not be seen as remainder terms contrary to the functions g_0 and g_1. Let us introduce the function

$$\Lambda : (\hat{\rho}, \hat{u}, \hat{a}, \hat{\omega}) \to (\tilde{\rho}, \tilde{u}, a, \omega).$$

We prove the three following results for R small enough:

1. The map Λ is well defined on a ball of radius R in $X_1 \times X_2 \times H^3(0,\tau) \times H^2(0,\tau)$
2. It takes values in the same ball of radius R
3. It is continuous for the topology of the space $L^\infty(0,\tau;H^1(\Omega_F(0))) \times L^2(0,\tau;H^2(\Omega_F(0)))^3 \times H^2(0,\tau) \times H^1(0,\tau)$.

To prove the two first results, for a start, we establish estimates on $\tilde{\rho}$ which is seen as a solution of a transport equation with a right-hand side depending on $\tilde{u}$. Then, we simultaneously obtain estimates on $\tilde{u}$, a and ω by considering that $\tilde{u}$ is the solution of a parabolic equation with a right-hand side depending on $\tilde{\rho}$. Thanks to energy estimates on the system satisfied by $\tilde{u}$, a and ω and the system satisfied by the time derivatives of $\tilde{u}$, a and ω, we get the desired estimates in spaces which are regular in time. The space regularity on the fluid velocity is then derived thanks to regularity results for elliptic equations.

Since the space $L^\infty(0,\tau;H^1(\Omega_F(0))) \times L^2(0,\tau;H^2(\Omega_F(0))) \times H^2(0,\tau) \times H^1(0,\tau)$ is compact in $X_1 \times X_2 \times H^3(0,\tau) \times H^2(0,\tau)$, these three points allow to use Schauder fixed point theorem and to deduce from it the existence of a fixed point. The uniqueness of the local solution is then proved independently with the help of an energy estimate on the system satisfied by the difference between two solutions.

The second intermediate result gives a global estimate on a solution of the problem. More precisely, we suppose that the problem admits a solution defined on $(0,T)$ for some arbitrary T. Then there exist $\delta > 0$ and C_2 independent of T such that, if $N_T(\rho - \overline{\rho}, u, a, \omega) \leqslant \delta$, we have

$$N_T(\rho - \overline{\rho}, u, a, \omega) \leqslant C_2(\|\rho_0 - \overline{\rho}\|_{H^3(\Omega_F(0))} + \|u_0\|_{H^3(\Omega_F(0))} + |a_0| + |\omega_0|).$$

The proof of this inequality is very complex and is inspired by the works by Matsumura and Nishida [MN82] and [MN80] which consider a compressible fluid alone. It relies on the combination of several regularity estimates in time and space, inside the domain and on the boundary. Contrary to the first result, we stay on the moving domain for the fluid equations and we rewrite the equations by putting in the right-hand side remainder terms. Let us define $\rho^* = \rho - \overline{\rho}$. If we divide the equation (1.2) by ρ, we can rewrite the fluid equations:

$$\partial_t \rho^* + u \cdot \nabla \rho^* + \overline{\rho} \nabla \cdot u = f_0(\rho^*, u)$$

and

$$\partial_t u - \frac{1}{\overline{\rho}} \nabla \cdot (2\mu \epsilon(u) + \mu'(\nabla \cdot u)\mathrm{Id}) + p^0 \nabla \rho = f_1(\rho^*, u)$$

where f_0 and f_1 are (at least) quadratic terms with respect to ρ^* and u.

To make the estimates come through in order to get our global estimate, the choice in the rewriting of the equations plays a key role. In the first equation, we keep the quadratic term $u \cdot \nabla \rho^*$ in the left-hand side in order to have the total derivative of ρ^*. Linearizing this term leads to problems in the estimates because of the spatial derivative of ρ^*. Moreover, the pressure law is linearized around $\overline{\rho}$. If we take a linear combination of the two equations (or of their derivatives) by multiplying the first equation by $\dfrac{p^0}{\overline{\rho}}$ and by adding the second equation, we are able to deal with the terms in the estimate which come from $\overline{\rho} \nabla \cdot u$ and $p^0 \nabla \rho$: their main parts cancel thanks to an integration by parts.

2.2 Interaction between a compressible fluid and an elastic structure [BG10]

2.2.1 Presentation of the problem

We consider the same geometry as in the previous study but in the work [BG10] also made with Sergio Guerrero, we assume that the structure is elastic and that its evolution is described by the linearized elasticity equation. As briefly mentioned in the introduction, this leads to many additional difficulties.

A first difficulty is linked to the regularity of the domains occupied by the structure and by the fluid at time t. If we denote by ξ the elastic displacement defined on the solid domain $\Omega_S(0)$, the flow is given by:

$$\chi_S(t,y) = y + \xi(t,y). \tag{1.25}$$

To start with, let us consider the spaces corresponding to the regularity given by the a priori energy estimate. Then we have the following regularity:

$$\sqrt{\rho}u \in L^\infty_T(L^2), \quad u \in L^2_T(H^1), \quad \xi \in W^{1,\infty}(0,T;L^2(\Omega_S(0))) \cap L^\infty(0,T;H^1(\Omega_S(0)))$$

where we have taken again the notations (1.20) for the regularity of u. These spaces correspond to the usual energy spaces for the compressible Navier-Stokes equations and for the linearized elasticity equations taken one at a time.

Thus we see that, at a given time t, the function $\chi_S(t,\cdot)$ will only belong to $H^1(\Omega_S(0))$. Next, if we define the set $\Omega_S(t) = \chi_S(t,\Omega_S(0))$ and the set $\Omega_F(t) = \Omega \setminus \overline{\Omega_S(t)}$ where the Navier-Stokes equations are set, we see that the regularity of χ_S is much too weak to get a problem defined in a classical way. In particular, the sets $\Omega_S(t)$ and $\Omega_F(t)$ are a priori not connected and we have much less than the Lipschitz regularity which allows to set the equations on classical domains (even if, in some cases, we can work with domains which do not have the Lipschitz regularity as in the works [CDEG05] and [Gra08]). In all cases, we at least need the spatial continuity of the flow at a given time. Thus, the energy spaces are not strong enough to give a sense to this problem.

It is also necessary to have a solid deformation regular enough to avoid collisions between the structure and the boundary of the cavity at least during a small time. To do so, we need an estimate on the velocity of the solid deformation in a norm $L^p(0,T;L^\infty(\Omega_S(0))$ with $p > 1$. At last, it will be necessary to avoid during a small time two physical situations which are not realistic: interpenetration (when two parts of the solid merge) and the loss of orientation of the solid.This will be possible if one has an estimate of the gradient of the solid deformation in the norm $L^\infty((0,T)\times\Omega_S(0))$. Having an estimate in this norm will also allow to invert the flow and to pass from an eulerian point of view to a lagrangian point of view.

Thus, for the problem that we consider, it does not seem possible to determine a solution (ρ,u,ξ) in the energy space. To deal with more regular solutions, we have two options.

The first possibility consists of regularizing the problem satisfied by the elastic deformation by adding artificial regularity. It is used with different variants in several papers. For instance, in the paper [Bou05], achieved during my PhD, a regularizing term of order 6 is added in the elasticity equation. For this problem, the existence of weak solution is proved with an elastic deformation belonging to the space $H^1(0,T;H^3(\Omega_S(0)))$ thanks to the regularizing term. The techniques which are used to study the fluid equations come from the works by E. Feireisl and his co-authors [FNP01], [Fei01] and by P.-L. Lions [Lio96]. Even if this regularizing term is added in an ad hoc fashion, the theory of multipolar materials justifies from a physical point of view how high-order spaces derivatives of the deformation velocity may appear in the stress tensor. The regularization that I considered in [Bou05] corresponds to a tripolar material. The same regularizing term is considered in [Bou07] (elastic structure in an incompressible fluid). We also refer to [CDEG05], [BdV04] and [MC13] (elastic plate which occupies a part of the boundary of the fluid domain) where regularizing terms of different types are added in the elasticity or plate equation. At last, in [DEGLT01] and in [BST12] presented in Subsection 2.3, it is assumed that the elastic deformation belongs to a finite-dimension space, which allows to get an elastic deformation regular in space. By this way, the problem corresponds to a coupling between the fluid equation and a system of ODE as in the case of an interaction with a rigid structure.

The second possibility consists of considering the classical linearized elasticity equation and to show the existence of strong solution so that the elastic deformation will satisfy the desired regularity without adding a regularizing term in the elasticity equation. That is the way we have proceeded in [BG10] where we obtain that the elastic deformation has the following regularity:

$$\xi \in C^0([0,T]; H^3(\Omega_S(0))) \cap C^3([0,T]; L^2(\Omega_S(0)))$$

and thus the deformation belongs to $C^1([0,T]; H^2(\Omega_S(0)))$.

There are few results of existence of strong solutions without regularizing terms in the elasticity equation. For an incompressible fluid, we can quote [CS05] and [CS06] which considers a nonlinear elasticity law. The result in the first paper has been improved in [KT12a] and in [RV14] for a particular geometry in $\mathbb{R}^3$: at initial time, the fluid fills a domain of the type $\mathbb{R}^2 \times \big(]0, m_1[\cup]m_2, m_3[\big)$ and the solid fills a domain of the type $\mathbb{R}^2 \times]m_1, m_2[$. For a compressible fluid, [BG10] gives the first result of existence of strong solution without regularization on the elastic deformation. Since this paper, [KT12b] has improved our result by considering less regular solutions (we will compare more precisely the results in paragraph 2.2.2). Let us also mention that the paper [Gra08] shows the existence of weak solution for the interaction between an incompressible fluid and a plate which occupies a part of the boundary of the fluid domain: for this geometry, the regularity obtained at the energy level is enough to tackle the difficulties linked to moving domains.

To obtain a strong regularity on the solution, the difficulties come from the fact that the coupling links up equations of different nature. By this way, there is a mismatch between the regularity coming from the hyperbolic equation satisfied by ξ and the regularity coming from the equation of parabolic type (if we look at the linearized equation) satisfied by u. This leads to a substantial difficulty to choose the space where we will look for our solution. In some cases, it can be taken advantage of this mismatch: for instance, in [Gra08], the elastic deformation is more regular thanks to the coupling condition with the fluid.

Let us now describe the model which has been considered in [BG10]. We take again the same notations as in the previous subsection. The velocity u and the density ρ are solution of the compressible Navier-Stokes equations (1.1) and (1.2) where the tensor $\sigma(u,p)$ is given by (1.10) and the pressure is given by:

$$p = P(\rho) - P(\rho^0). \tag{1.26}$$

Contrary to the case of a rigid structure where the pressure acts up to a constant, this constant is now fixed. In the definition of p, $\rho^0 \in \mathbb{R}_+^*$ is a fixed value and $\rho = \rho^0$, $u = 0$ and $\xi = 0$ is a stationary solution of the problem. We assume that P is of class C^∞ in $\mathbb{R}_+^*$.

The elastic deformation is solution of the linearized elasticity equation:

$$\partial_{tt}\xi(t,y) - \nabla \cdot \big(2\lambda\epsilon(\xi) + \lambda'(\nabla\cdot\xi)\mathrm{Id}\big)(t,y) = 0, \ \forall\,(t,y) \in (0,T)\times\Omega_S(0) \tag{1.27}$$

where the Lamé constants λ and λ' satisfy the conditions:

$$\lambda > 0, \ \ \lambda' \geqslant 0. \tag{1.28}$$

We have considered, without loss of generality, that the solid density is equal to 1.

To express the coupling conditions between the fluid and the structure, we need to introduce a flow which allows to pass from an eulerian point of view to a lagrangian point of view. This flow will be used in the resolution of the problem and will allow to express the fluid equations given in $\Omega_F(t)$ on the fixed domain $\Omega_F(0)$.

A first way to define the flow is to use the same method as in the previous subsection: it consists in considering the flow on the solid domain given by (1.25) and to extend it on the fluid domain. Here we choose another method which consists of defining the flow from the fluid velocity. More precisely, for all $y \in \Omega_F(0)$, we define the flow $\chi(\cdot, y)$ as the solution of the ordinary differential equation:

$$\begin{cases} \partial_t \chi(t,y) = u(t, \chi(t,y)) \\ \chi(0,y) = y. \end{cases} \tag{1.29}$$

By this way, the flow corresponds to the trajectories of the fluid particles. This choice in the definition of the flow allows to get a more regular flow.

The motions of the fluid and the solid are coupled at the interface. The first condition expresses the fact that the velocity at the interface is continuous and the second condition enforces the continuity of the normal component of the stress tensor. We have the following conditions: on $(0,T) \times \partial\Omega_S(0)$

$$\begin{cases} u \circ \chi = \partial_t \xi \\ \big(2\mu\epsilon(u) + \mu'(\nabla\cdot u)\mathrm{Id} - (P(\rho) - P(\rho^0))\mathrm{Id}\big) \circ \chi \,\mathrm{cof}\,\nabla\chi\, n = \big(2\lambda\epsilon(\xi) + \lambda'(\nabla\cdot\xi)\mathrm{Id}\big) n. \end{cases} \tag{1.30}$$

The system is completed by Dirichlet conditions on the external boundary:

$$u = 0 \text{ on } (0,T) \times \partial\Omega. \tag{1.31}$$

To end with, we fix the initial conditions:

$$\rho(0,\cdot) = \rho_0 \text{ in } \Omega_F(0),\ u(0,\cdot) = u_0 \text{ in } \Omega_F(0) \tag{1.32}$$

and

$$\xi(0,\cdot) = \xi_0 \text{ in } \Omega_S(0),\ \partial_t\xi(0,\cdot) = \xi_1 \text{ in } \Omega_S(0). \tag{1.33}$$

2.2.2 Description of the result obtained in [BG10]

In [BG10], we have proved the local in time existence and uniqueness of regular solutions. More precisely, we have considered initial conditions which satisfy:

$$\rho_0 \in H^3(\Omega_F(0)),\ \rho_0 \geqslant \rho_{min} > 0 \text{ in } \Omega_F(0),\ u_0 \in H^4(\Omega_F(0)),\ \xi_0 \in H^3(\Omega_S(0)),\ \xi_1 \in H^2(\Omega_S(0)) \tag{1.34}$$

and we have assumed that the initial conditions satisfy compatibility conditions that we do not specify here. We have then proved the following result:

Theorem 1.4 *Let $(\rho_0, u_0, \xi_0, \xi_1)$ satisfy (1.34). There exists a time $T^* > 0$ such that the system of equations (1.1), (1.2), (1.26), (1.27) completed by the boundary conditions (1.30)-(1.31) and the* *initial conditions (1.32)-(1.33) admits a unique solution* (ρ, u, ξ) *defined in* $(0, T^*)$ *in the space* $Y_{T^*} := Y^1_{T^*} \times Y^2_{T^*} \times Y^3_{T^*}$ *where*

$$Y^1_{T^*} := L^2_{T^*}(H^2) \cap C^0_{T^*}(H^{7/4}),$$

$$Y^2_{T^*} := L^2_{T^*}(H^3) \cap C^0_{T^*}(H^{11/4}) \cap H^2_{T^*}(H^1) \cap C^2_{T^*}(L^2),$$

$$Y^3_{T^*} := C^0([0,T^*]; H^3(\Omega_S(0))) \cap C^2([0,T^*]; H^1(\Omega_S(0))) \cap C^3([0,T^*]; L^2(\Omega_S(0))).$$

Moreover, there exists an increasing positive function f *depending on* $1/\rho_{min}$, $\|\rho_0\|_{\Omega_F(0)}$, $\|u_0\|_{H^4(\Omega_F(0))}$, $\|\xi_0\|_{H^3(\Omega_S(0))}$ *and* $\|\xi_1\|_{H^2(\Omega_S(0))}$ *such that*

$$\|(\rho, u, \xi)\|_{Y_{T^*}} \leqslant f(1/\rho_{min}, \|\rho_0\|_{\Omega_F(0)}, \|u_0\|_{H^4(\Omega_F(0))}, \|\xi_0\|_{H^3(\Omega_S(0))}, \|\xi_1\|_{H^2(\Omega_S(0))}).$$

Here, we have taken again the definitions (1.20) of the spaces in domains depending on time.

We notice that the initial condition for the fluid velocity u_0 belongs to $H^4(\Omega_F(0))$ while, for our solution u, at the final time T^*, we only get that $u(T^*, \cdot)$ belongs to $H^{11/4}(\Omega_F(T^*))$. Thus, it is not possible to extend the solution by starting again from T^* and by applying again our local result.

This illustrates a loss of regularity in space for the fluid velocity which comes from the hyperbolic-parabolic coupling. Roughly speaking, if we differentiate two times a wave-like hyperbolic equation and if we look at the energy estimate, we have the following regularity:

$$\xi \in C^2(0, T; H^1) \cap C^3(0, T; L^2).$$

Then, by applying classical results for the regularity of elliptic equations, we thus expect to get a function in $C^0(0, T; H^3)$, and that is what we get on the coupled problem that we consider.

In the same way, if we differentiate two times a heat-like parabolic equation and if we look at the energy estimate, we have the following regularity:

$$u \in H^2(0, T; H^1) \cap C^2(0, T; L^2).$$

Thus we expect to get a function in $L^2(0, T; H^5) \cap C^0(0, T; H^4)$. We then see that we do not reach this maximal regularity for the solution u of the coupled problem. This comes from the fact that u satisfies boundary conditions which involve the function ξ which is less regular in space. The parabolic regularity is thus restrained by the coupling with the hyperbolic equation.

Remark 1.5 *Let us notice that, in the case of incompressible Navier-Stokes equations, the paper by Coutand and Shkoller [CS05] shows the existence of solutions in similar spaces but for initial conditions which are still more regular: existence is obtained for* u_0 *in* $H^5(\Omega_F(0))$ *and uniqueness for* u_0 *in* $H^7(\Omega_F(0))$.

If we compare the result given by Theorem 1.4 to the results for a compressible fluid alone (for instance, in [MN82], the regularity of ρ and u is given by (1.15)), we see that we need to consider more regular solutions. In particular, our result gives an estimate of the second derivative in time of u which is obtained by considering the system differentiated two times in time.

This result has since been improved by Kukavica and Tuffaha in [KT12b]. In their work, the space the solution belongs to is less regular. Let us quote more precisely their result: for $r \in (0, (\sqrt{2} - 1)/2)$, they assume that the initial conditions satisfy:

$$\rho_0 \in H^{3/2+r},\ u_0 \in H^3(\Omega_F(0)),\ \xi_0 \in H^{5/2+r}(\Omega_S(0)),\ \xi_1 \in H^{3/2+r}(\Omega_S(0)).$$

In this framework, they obtain that the solution belongs to the following spaces:

$$\rho \in C^2_{T^*}(L^2) \cap C^1_{T^*}(H^{3/2+r}),$$
$$u \in H^1_{T^*}(H^{3/2+r}) \cap C^0_{T^*}(H^{5/2+r}) \cap H^2_{T^*}(L^2) \cap C^1_{T^*}(H^1),$$
$$\xi \in C^0([0,T^*]; H^{5/2+r}(\Omega_S(0))) \cap C^1([0,T^*]; H^{3/2+r}(\Omega_S(0))).$$

The solutions that are considered in [KT12b] are thus less regular than in Theorem 1.4 and the regularity in time of the fluid velocity is similar to the one for a compressible fluid alone in [MN82]. An important step in their proof is to show estimates on the trace of the elastic deformation, these estimates are obtained thanks to hidden regularity results ([LLT86]). Let us notice that, in their result, there is also a loss of regularity for the fluid velocity (at time T^*, the velocity only belongs to $H^{5/2+r}$) but the discrepancy between the regularity of u_0 and of $u(T^*)$ is smaller than in our case.

One of the interests of our result compared to the one in [KT12b] is that it gives the existence of solutions in more usual Sobolev spaces. Moreover, our proof is self-contained and, even it is very technical, it relies on a scheme relatively simple. In some sense, the proofs for a compressible fluid alone given in [MN82] and for a compressible fluid with a rigid structure [BG09] are more technical than the proof of Theorem 1.4.

2.2.3 Scheme of the proof

The first step consists in doing a partial linearization of the problem with the help of a given fluid velocity. More precisely, we introduce the space $X_{T,R}$ defined by:

$$X_{T,R} = \{v \in Y_{p,q},\ v = 0 \text{ on } (0,T)\times\partial\Omega,\ v(0) = u_0 \text{ in } \Omega_F(0) \text{ and } N_T(v) \leqslant R\},$$

where, for $1 < p < 2$ and $4 < q < \infty$

$$Y_{p,q} = W^{2,p}(0,T; H^1(\Omega_F(0))) \cap W^{2,q}(0,T; L^2(\Omega_F(0))) \cap L^q(0,T; H^{11/4}(\Omega_F(0))) \cap L^p(0,T; H^3(\Omega_F(0)))$$

and N_T is the natural norm associated to $Y_{p,q}$.

Let $\hat{v}$ be a given lagrangian velocity in $X_{T,R}$ and let us define the flow associated to $\hat{v}$ by:

$$\hat{\chi}(t,y) = y + \int_0^t \hat{v}(s,y)\,ds \quad \forall y \in \Omega_F(0).$$

This function belongs to $W^{1,q}(0,T; H^{11/4}(\Omega_F(0)))$. Thus, the function $y \mapsto \hat{\chi}(t,\cdot)$ is invertible for R small enough since $H^{11/4} \hookrightarrow C^1$ in dimension 3. This allows to define the eulerian velocity by:

$$\hat{u}(t,x) := \hat{v}(t, \hat{\chi}^{-1}(t,x)),\ \forall t \in (0,T),\ \forall x \in \widehat{\Omega}_F(t)$$

where $\widehat{\Omega}_F(t)$ is given by $\widehat{\Omega}_F(t) = \hat{\chi}(t, \Omega_F(0))$.

Now, we introduce the following problem

$$\begin{cases} \big(\partial_t \rho + \nabla\cdot(\rho \hat{u})\big)(t,x) = 0,\ \forall\, x \in \widehat{\Omega}_F(t), \\ \big(\rho\partial_t u + \rho(\hat{u}\cdot\nabla)u\big)(t,x) - \nabla\cdot\big(2\mu\epsilon(u) + \mu'(\nabla\cdot u)\mathrm{Id} - (P(\rho) - P(\rho^0))\mathrm{Id}\big)(t,x) = 0,\ \forall\, x \in \widehat{\Omega}_F(t) \end{cases} \tag{1.35}$$

completed by the elasticity equation (1.27) and by the coupling conditions on $(0,T)\times\partial\Omega_S(0)$:

$$\begin{cases} u \circ \hat{\chi} = \partial_t \xi, \\ \big(2\mu\epsilon(u) + \mu'(\nabla\cdot u)\mathrm{Id} - (P(\rho) - P(\rho^0))\mathrm{Id}\big) \circ \hat{\chi}\, \mathrm{cof}\,\nabla\hat{\chi}\, n = \big(2\lambda\epsilon(\xi) + \lambda'(\nabla\cdot\xi)\mathrm{Id}\big)n. \end{cases}$$

This system is still nonlinear since the second equation of (1.35) is not linearized with respect to the density. We notice that, thanks to the linearization, the equation satisfied by ρ is uncoupled from the other equations. By this way, we will easily prove that the first equation of (1.35) admits a unique regular solution ρ and the pressure term in the second equation of (1.35) will be viewed as a right-hand side. Let us make the following change of variables:

$$v(t,y) = u(t,\hat{\chi}(t,y)),\ \gamma(t,y) = \rho(t,\hat{\chi}(t,y)) - \rho^0,\ \text{for all } (t,y) \in (0,T) \times \Omega_F(0).$$

Then, the equations become:

$$\partial_t \gamma + \gamma(\nabla \hat{v} (\nabla \hat{\chi})^{-1} : \mathrm{Id}) + \rho^0 (\nabla \hat{v} (\nabla \hat{\chi})^{-1} : \mathrm{Id}) = 0 \text{ in } (0,T) \times \Omega_F(0). \tag{1.36}$$

$$\begin{aligned}(\rho^0 + \gamma) \det \nabla \hat{\chi} \partial_t v - \nabla \cdot \big[\big(\mu(\nabla v (\nabla \hat{\chi})^{-1} + (\nabla \hat{\chi})^{-T} \nabla v^T) + \mu' (\nabla v (\nabla \hat{\chi})^{-1} : \mathrm{Id})\mathrm{Id} \\ - (P(\rho^0 + \gamma) - P(\rho^0))\mathrm{Id}\big) \mathrm{cof}\, \nabla \hat{\chi}\big] = 0 \text{ in } (0,T) \times \Omega_F(0).\end{aligned} \tag{1.37}$$

We thus notice that this change of variables which involves the flow corresponding to the motion of the particles in the fluid simplifies the equations: if the space variable is fixed, the equation (1.36) becomes an ordinary differential equation and we can easily compute explicitly the solution γ. From this expression, we deduce that

$$\gamma \in W^{1,q}(0,T; H^{7/4}(\Omega_F(0))) \cap W^{2,q}(0,T; L^2(\Omega_F(0))) \cap L^2(0,T; H^2(\Omega_F(0))).$$

Moreover, in the second equation (1.37), there is no more convective term.

Our system is completed by boundary conditions which are written on $(0,T) \times \partial\Omega_S(0)$:

$$\begin{cases} v = \partial_t \xi, \\ \big(\mu(\nabla v (\nabla \hat{\chi})^{-1} + (\nabla \hat{\chi})^{-T} \nabla v^T) + \mu' (\nabla v (\nabla \hat{\chi})^{-1} : \mathrm{Id})\mathrm{Id} - (P(\rho^0 + \gamma) - P(\rho^0))\mathrm{Id}\big) \mathrm{cof}\, \nabla \hat{\chi}\, n \\ = \big(2\lambda \epsilon(\xi) + \lambda' (\nabla \cdot \xi)\mathrm{Id}\big) n. \end{cases}$$

Our goal is now to show that

$$\begin{aligned} & v \in H^2(0,T; H^1(\Omega_F(0))) \cap C^2(0,T; L^2(\Omega_F(0))) \cap C^0(0,T; H^{11/4}(\Omega_F(0))) \cap L^2(0,T; H^3(\Omega_F(0))) \\ & \text{and } \overline{N}_T(v) := \|v\|_{H^2(H^1)} + \|v\|_{C^2(L^2)} + \|v\|_{C^0(H^{11/4})} + \|v\|_{L^2(H^3)} \leqslant M \end{aligned} \tag{1.38}$$

for some $M > 0$. Since there exists $\alpha > 0$ such that

$$N_T(v) \leqslant T^\alpha \overline{N}_T(v)$$

we will deduce that v belongs to $X_{T,R}$ for T sufficiently small.

To prove the property (1.38), we proceed in two steps. First, we prove energy estimates on the system, the system differentiated in time one time and then two times. This allows to get simultaneously estimates on v in the norms $H^2(0,T; H^1(\Omega_F(0)))$ and $C^2(0,T; L^2(\Omega_F(0)))$ and on ξ in the norms $C^2([0,T]; H^1(\Omega_S(0)))$ and $C^3([0,T]; L^2(\Omega_S(0)))$ with respect to the initial conditions and remaining terms. To show the regularity in space of the solution, we use classical results of elliptic regularity. To do so, the global system is written as two coupled elliptic systems: first of all, $t \in (0,T)$ being fixed, v is solution of:

$$\begin{cases} -\nabla \cdot (2\mu \epsilon(v) + \mu' (\nabla \cdot v)\mathrm{Id}) = F & \text{in } \Omega_F(0), \\ (2\mu \epsilon(v) + \mu' (\nabla \cdot v)\mathrm{Id}) n = G & \text{on } \partial\Omega_S(0), \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

wher F and G are given by:

$$\begin{aligned} F &= -(\gamma+\rho^0)\det(\nabla\hat{\chi})\partial_t v - \nabla\cdot((P(\rho^0+\gamma)-P(\rho^0))\mathrm{cof}\,(\nabla\hat{\chi})) + F_R \\ G &= (\lambda(\nabla\xi+\nabla\xi^T)+\lambda'\nabla\cdot\xi)n + (P(\rho^0+\gamma)-P(\rho^0))\mathrm{cof}\,(\nabla\hat{\chi})n + G_R \end{aligned}$$

with remaining terms F_R and G_R that are not explicitly written here. The estimates already obtained on $\partial_t v$ and γ allow to get the following inequality: there exists $C > 0$ and $\kappa > 0$ such that:

$$\|v\|_{L^\infty(0,T;H^2(\Omega_F(0)))} \leqslant C + T^\kappa \overline{N}_T(v) + C\|\xi\|_{L^\infty(0,T;H^2(\Omega_S(0)))}. \tag{1.39}$$

The last term directly comes from the terms involving ξ in G.

In a similar way, we rewrite the elasticity equation: $t \in (0,T)$ being fixed, we have

$$\begin{cases} -\nabla\cdot(2\lambda\epsilon(\xi)+\lambda'(\nabla\cdot\xi)\mathrm{Id}) = -\partial_{tt}\xi & \text{in } \Omega_S(0), \\ \xi(t,\cdot) = \xi_0 + \displaystyle\int_0^t v & \text{on } \partial\Omega_S(0). \end{cases} \tag{1.40}$$

Using the estimates already obtained on $\partial_{tt}\xi$, we get the estimate: there exist $C > 0$ and $\kappa > 0$ such that:

$$\|\xi\|_{L^\infty(0,T;H^2(\Omega_S(0)))} \leqslant C + T^\kappa \overline{N}_T(v) + CT\|v\|_{L^\infty(0,T;H^2(\Omega_F(0)))}. \tag{1.41}$$

The last term directly comes from the term involving v in the boundary condition. In this estimate, it is capital to have T in front of the norm of v in $L^\infty(0,T;H^2(\Omega_F(0)))$ since, combining (1.39) and (1.41), we are able to deduce that, for T small enough:

$$\|v\|_{L^\infty(0,T;H^2(\Omega_F(0)))} + \|\xi\|_{L^\infty(0,T;H^2(\Omega_S(0)))} \leqslant C + T^\kappa \overline{N}_T(v).$$

To get more regularity in space on the solution, we use a bootstrap method: thanks to the estimates on v and ξ in $L^\infty(H^2)$, we can show that F et G are more regular and deduce estimates on v and ξ in $L^\infty(H^{11/4})$. And this last result allows to estimate v in $L^2(H^3)$ and ξ in $L^\infty(H^3)$. This allows to conclude that the property (1.38) is satisfied by v and thus v belongs to $X_{T,R}$ for T small enough.

We then conclude the proof with a fixed point argument. Let us consider the application:

$$\Lambda : \hat{v} \in X_{T,R} \to v \in X_{T,R}.$$

Then, we show that Λ is a contraction on $X_{T,R}$ for the norm in the space $L^2(H^2) \cap H^1(L^2)$. By this way, we get the existence and uniqueness of solutions in $X_{T,R}$ for our problem.

2.3 Interaction between an incompressible fluid and an elastic structure [BST12]

During my PhD, I have worked on the existence of weak solutions for an interaction problem between an elastic structure and an incompressible fluid modelled by the Navier-Stokes equations [Bou07]. The model of the structure considered in this work was introduced in [GMM07] and involves nonlinear coupled equations coming from the decomposition of the structure displacement of the structure in large rigid displacements and small elastic deformations.

As presented in paragraph 2.2.1, showing the existence of weak solution for the general problem is out of reach and it is necessary to regularize the elasticity equations. To do so, an option consists of considering an approximation of the linear elasticity equation by a finite-dimension system. This model is considered in [DEGLT01] and we have considered it again in [BST12]. In this paper, achieved with Erica Schwindt and Takéo Takahashi, we have worked on the existence of strong solution for this problem.

For the elastic structure, the model relies on a Galerkin approximation of the elastic deformation ξ which is written as a linear combination of a family of N_0 functions. As previously, we consider an elastic structure immersed in a fluid and the ensemble evolves in a cavity $\Omega \subset \mathbb{R}^3$. In the fluid, the equations are the incompressible Naver-Stokes equations:

$$\left\{ \begin{array}{ll} \partial_t u + (u \cdot \nabla) u - \nabla \cdot \sigma(u, p) = 0 & \text{in } \Omega_F(t) \\ \nabla \cdot u = 0 & \text{in } \Omega_F(t) \\ u(0, \cdot) = u_0 & \text{in } \Omega_F(0) \end{array} \right. \tag{1.42}$$

with

$$\sigma(u, p) = 2\nu\epsilon(u) - p\mathrm{Id}$$

where $\nu > 0$ is the kinematic viscosity of the fluid. This equation is completed by condition on the external boundary:

$$u = 0 \text{ on } \partial\Omega. \tag{1.43}$$

On the interface, the first condition enforces the continuity of the velocity: on $(0, T) \times \partial\Omega_S(0)$

$$u \circ \chi = \partial_t \xi \tag{1.44}$$

where the flow χ is given by

$$\chi(t, y) = y + \xi(t, y), \; \forall \, (t, y) \in (0, T) \times \Omega_S(0). \tag{1.45}$$

The second condition on the interface expresses the fact that the normal component of the stress tensor of the solid matches with the projection of the normal component of the stress tensor of the fluid on the family of the N_0 functions (this condition will appear in the right-hand side of the variational formulation (1.50)).

One may notice that the incompressibility property of the fluid adds a compatibility constraint on the deformation ξ. Indeed, the boundary conditions satisfied by the velocity enforce

$$\int_{\Omega_F(t)} \nabla \cdot u(t, x) \, dx = \int_{\partial\Omega_F(t)} u(t, x) \cdot n \, d\gamma_x = \int_{\partial\Omega_S(t)} u(t, x) \cdot n \, d\gamma_x = \int_{\partial\Omega_S(t)} \partial_t \xi(t, \chi^{-1}(t, x)) \cdot n \, d\gamma_x.$$

and thus, according to the incompressibility hypothesis:

$$\int_{\partial\Omega_S(t)} \partial_t \xi(t, \chi^{-1}(t, x)) \cdot n \, d\gamma_x = 0. \tag{1.46}$$

This condition expresses the fact that the volume of the solid can not vary since it evolves inside a fluid which occupies a constant volume and since the fluid-structure ensemble is isolated according to the null velocity condition on the boundary of the cavity.

Let us specify now the model which describes the elastic deformation. If we consider that ξ satisfies the linearized elasticity equation (1.27) and the coupling condition:

$$\sigma(u, p) \circ \chi \operatorname{cof} \nabla\chi \, n = \big(2\lambda\epsilon(\xi) + \lambda'(\nabla \cdot \xi)\mathrm{Id}\big) n \text{ on } (0, T) \times \Omega_S(0),$$

then ξ satisfies the following variational formulation: for all $\eta \in C^1([0,T];C^1(\Omega_S(0)))$

$$\begin{aligned}\int_{\Omega_S(0)} \partial_{tt}\xi \cdot \partial_t \eta \, dy + 2\lambda \int_{\Omega_S(0)} \varepsilon(\xi) : \varepsilon(\partial_t \eta) dy + \lambda' \int_{\Omega_S(0)} \nabla \cdot \xi \, \nabla \cdot \partial_t \eta \, dy \\ = \int_{\partial\Omega_S(t)} \sigma(u,p)(t,x) n_x \cdot \partial_t \eta(t, \chi^{-1}(t,x)) \; d\gamma_x.\end{aligned} \tag{1.47}$$

We will consider an approximation of this problem in finite dimension.

To do so, we introduce $\xi_1, \ldots, \xi_{N_0}$ ($N_0 \geqslant 1$) an orthonormal family in $L^2(\Omega_S(0))$ of elements in $H^3(\Omega_S(0))$ which satisfy, for all $1 \leqslant i \leqslant N_0$

$$\int_{\partial\Omega_S(0)} \xi_i(y) \cdot n_y \, d\gamma_y = 0.$$

If ξ is a linear combination of $\xi_1, \ldots, \xi_{N_0}$, it has a priori no reason to satisfy (1.46). We thus modify the writing of ξ by adding in the linear combination a term whose role is to correct the volume variations by retracting or dilating the structure. To do so, we take as additional function in the basis a lifting $\xi_0 \in H^3(\Omega_S(0))$ of the external normal on $\partial\Omega_S(0)$. Then, we show that there exists $r_1 > 0$ and a function $\phi : B(0, r_1) \subset \mathbb{R}^{N_0} \to \mathbb{R}$ such that, for all function $(\alpha_1, \ldots, \alpha_{N_0}) \in C^1([0,T]; B(0,r_1))$, the associated deformation

$$\xi(t,y) := \sum_{i=0}^{N_0} \alpha_i(t) \xi_i(y), \quad \text{with} \quad \alpha_0(t) = \phi(\alpha_1(t), \ldots, \alpha_{N_0}(t)), \tag{1.48}$$

satisfies (1.46). Using this construction which is presented in [DEGLT01] and [Bou07], we can show that the derivative in time of ξ can be written:

$$\partial_t \xi(t,y) = \sum_{i=1}^{N_0} \beta_i(t) \widehat{\xi}_i(t,y), \quad \text{with} \quad \beta_i = \alpha_i'$$

where the functions $\widehat{\xi}_i$ are linear combinations of ξ_i and ξ_0, for all $1 \leqslant i \leqslant N_0$ which satisfy

$$\int_{\partial\Omega_S(t)} \widehat{\xi}_i(t, \chi^{-1}(t,x)) \cdot n_x \, d\gamma_x = 0. \tag{1.49}$$

We can now specify the problem satisfied by ξ: we look for ξ given by the linear combination (1.48) and solution of

$$\begin{aligned}\int_{\Omega_S(0)} \partial_{tt}\xi \cdot \widehat{\xi}_i dy + 2\lambda \int_{\Omega_S(0)} \varepsilon(\xi) : \varepsilon(\widehat{\xi}_i) dy + \lambda' \int_{\Omega_S(0)} \nabla \cdot \xi \, \nabla \cdot \widehat{\xi}_i dy \\ = \int_{\partial\Omega_S(t)} \sigma(u,p)(t,x) n_x \cdot \widehat{\xi}_i(t, \chi^{-1}(t,x)) \; d\gamma_x, \quad i = 1, \ldots, N_0\end{aligned} \tag{1.50}$$

This problem is completed by initial conditions. We assume that the initial configuration of the solid corresponds to its reference configuration. By this way

$$\xi(0,\cdot) = 0 \text{ in } \Omega_S(0) \text{ and } \partial_t \xi(0,\cdot) = \xi^1 \text{ in } \Omega_S(0) \tag{1.51}$$

with ξ^1 given by $\xi^1 = \sum_{i=1}^{N_0} \beta_i^0 \xi_i$.

Remark 1.6 *As explained in [Gra08], if we consider the full problem (with (1.47) instead of (1.50)), the fluid pressure is not defined up to a constant but in a unique way. In the case of the appoximated problem that we consider, the pressure is defined up to a constant. Indeed, since the* $\widehat{\xi}_i$ *satisfy the condition (1.49), we see that, if* p *is solution of the problem, thus* $p + C$ *is also solution of the problem. This constant does not modify the elastic deformation since this one is constructed in such a way that the volume of the structure is preserved.*

Let us use again the notations (1.20) for the definition of the spaces in the fluid domain with a diffeomorphism χ in $H^2(0, T; H^3(\Omega_F(0)))$. We show in [BST12] the following result:

Theorem 1.7 *Suppose that* $u_0 \in H^1(\Omega_F(0))$ *and* ξ^1 *satisfy the following conditions:*

$$\begin{cases} \xi^1 = \displaystyle\sum_{i=1}^{N_0} \beta_i^0 \xi_i \\ \nabla \cdot u_0 = 0 & \text{in } \Omega_F(0) \\ u_0 = 0 & \text{on } \partial\Omega \\ u_0 = \xi^1 & \text{on } \partial\Omega_S(0). \end{cases} \tag{1.52}$$

Assume that, at the initial time, the solid does not touch the boundary of the cavity :

$$d(\Omega_S(0), \partial\Omega) > 0.$$

Then, there exists a time $T > 0$ *such that the system* (1.42), (1.43), (1.44), (1.50) *and* (1.51) *admits a unique solution (up to a constant for* p*)*

$$u \in L^2_T(H^2) \cap C^0_T(H^1) \cap H^1_T(L^2), \quad p \in L^2_T(H^1),$$
$$\alpha_1, \ldots, \alpha_{N_0} \in H^2(0, T).$$

The existence of weak solution for our system has been proved in [DEGLT01]. If we compare our existence result of strong solution to the one obtained in [CS05] which considers the whole problem without any regularization of the elasticity equation, the interest of our work is that we consider the same spaces as for the strong solutions of the incompressible Navier-Stokes equations. In particular, there is no loss of regularity on the solution with respect to the initial conditions.

For studies on the existence of solutions in the case of a rigid structure, we can quote the works [CSMHT00, Fei03b, GLS00, SMST02, GS09] which study the existence of weak solution and [CT08, Tak03, GS09] which study the existence of strong solution. In both cases, the spaces the solution belongs to are similar to the spaces for a fluid alone modeled by the incompressible Navier-Stokes equations.

Here, as in the case of a rigid structure, we look for a structure velocity in a space of finite dimension. However, there are several additional difficulties: the velocity of the deformation ξ is a linear combination of functions, the $\widehat{\xi}_i$, $1 \leqslant i \leqslant N_0$, which are defined in an implicit way since the condition (1.49) depends on the flow χ which is itself dependent on ξ. Thus the space where we look for the solution of our Galerkin problem (1.50) depends on the solution. The change of variables to bring back the fluid equations in a fixed domain is also more technical. In the case of a rigid structure, we can simply extend the translation and the rotation to the fluid domain. As detailed in [Tak03], it is possible to choose in a relatively simple way an extension of the flow defined on the structure such that the associated velocity is divergence free and this choice simplifies the change of variables. Here, we are going to extend the flow defined on $\Omega_S(0)$ by (1.45) to the domain Ω by:

$$\chi(t, y) = y + \mathcal{E}(\xi(t, y)), \ \forall\ (t, y) \in (0, T) \times \Omega$$

where $\mathcal{E}$ is a continuous and linear extension operator from $H^3(\Omega_S(0))$ in $H^3(\Omega) \cap H^1_0(\Omega)$ which is such that, for all $w \in H^3(\Omega_S(0))$,

$$\mathcal{E}(w) \text{ has a support in } \{y \in \Omega / d(y, \Omega_S(0)) < \epsilon\} \quad \text{with } 0 < \epsilon < d(\Omega_S(0), \partial\Omega).$$

Since we start from the reference configuration, the estimates on the solution allow to show that the elastic deformation will stay small in norm $L^\infty(W^{1,\infty})$ during some time and this allows to show that the flow χ is invertible in the whole domain Ω. Then, we introduce the change of variables, for all $t \in (0,T)$,

$$\begin{array}{lll} v(t,y) & = \det(\nabla\chi(t,y))\ (\nabla\chi(t,y))^{-1}\ u(t,\chi(t,y)), & \forall\, y \in \Omega_F(0) \\ q(t,y) & = \det(\nabla\chi(t,y))\ p(t,\chi(t,y)), & \forall\, y \in \Omega_F(0) \\ V(t,y) & = \det(\nabla\chi(t,y))\ (\nabla\chi(t,y))^{-1}\partial_t\xi(t,y), & \forall\, y \in \Omega_S(0). \end{array}$$

The definition of v corresponds to the Piola-Kirchhoff transform of u with the flow χ which is a common tool in the elasticity theory. It has the property to preserve the divergence null. More precisely, we have

$$\nabla_y \cdot v(t,y) = \det \nabla\chi(t,y)\ (\nabla_x \cdot u)(t,\chi(t,y)),\ \forall\,(t,y) \in (0,T) \times \partial\Omega_F(0).$$

The change of unknowns from ξ to V allows to get a simple expression of the coupling condition:

$$v(t,y) = V(t,y),\ \forall\,(t,y) \in (0,T) \times \partial\Omega_S(0).$$

Using this change of variables, we then get nonlinear coupled equations with variable coefficients written on a fixed domain. We write the system in the form:

$$Z' = AZ + R(Z)$$

where $Z = (v, \beta_1, \ldots, \beta_{N_0})$. In the operator R, we put all the terms involving the coefficients coming from the change of variables and the nonlinear terms coming from the Navier-Stokes equations. We then consider the linear system associated to our problem:

$$Z' = AZ + F.$$

The analysis of this system is done in a similar way as in [Tak03] through the study of the associated semi-group. To move on the nonlinear problem, we then use the Banach fixed point theorem on the linearization operator. To do so, we prove estimates on $R(Z)$ in order to show that the operator is defined from a ball in itself for T small enough and we prove estimates on the difference $R(Z^1) - R(Z^2)$ in order to show that the operator is a contraction for T small enough.

2.4 Conclusion

Through the presentation of the previous results, I have tried to shed light on the main difficulties raised by the questions of existence, uniqueness and regularity of solutions for fluid-structure interaction problems. There are still many open questions, especially when the structure is elastic and the existence of strong solutions for the interaction between an incompressible fluid and an elastic structure is still a very challenging topic as specified for instance in the introduction of [RV14].

In a work in progress, I am considering with Sergio Guerrero the interaction between a compressible fluid and an elastic structure described by a nonlinear law. In this work, we need to consider solutions which are more regular than in [BG10] (one level higher for the time regularity).

3 Controllability - [BO08], [BG13]

In this part, I present the works [BO08] and [BG13] which deal with the controllability of interaction problems between a viscous incompressible fluid and a rigid structure. The problems and the results which are considered in these two papers are very similar but there are significant differences in the considered methods.

In both cases, we consider a fluid modeled by the incompressible Navier-Stokes equations and a rigid structure which evolves inside the fluid. The motion of the structure is described by the laws of reciprocal actions. We assume that we can act on the fluid-structure ensemble by applying a force, that we call a control, on a subdomain arbitrary small of the fluid domain.

We then show that we can locally control the fluid velocity and the position and the velocity of the structure. In the first paper [BO08] done with Axel Osses, we assume that we are in dimension 2 and that the solid satisfies a symmetry condition which will be given later. The second paper [BG13] done with Sergio Guerrero generalizes the previous result: we are in dimension 3 and consider general shapes for the solid without symmetry condition anymore. The initial conditions are also less regular.

3.1 Presentation of the models

The fluid-structure ensemble is contained in a fixed bounded cavity $\Omega \subset \mathbb{R}^d$ with $d = 2$ or 3. We denote by $\mathcal{O} \subset\subset \Omega$ the domain where the control force acts. We suppose that the domains $\Omega_S(0)$ and Ω are regular (for instance C^4) and that

$$\Omega_S(0) \subset \Omega \setminus \mathcal{O}, \ d(\partial(\Omega \setminus \mathcal{O}), \overline{\Omega_S(0)}) \geqslant \delta_0 > 0. \tag{1.53}$$

As in Subsection 2.3, we consider the incompressible Navier-Stokes equations but we now have a right-hand side which corresponds to a force acting on $\mathcal{O}$: $\forall\, t > 0$, $\forall\, x \in \Omega_F(t)$

$$\left\{ \begin{array}{l} (\partial_t u + (u \cdot \nabla)u)(t, x) - \nabla \cdot \sigma(u, p)(t, x) = f(t, x)\zeta(x) \\ \nabla \cdot u(t, x) = 0. \end{array} \right. \tag{1.54}$$

Here, $\zeta \in C^2_c(\mathcal{O})$ satisfies $\zeta = 1$ in $\widetilde{\mathcal{O}} \subset\subset \mathcal{O}$ and f is the control force which acts on the system in $\mathcal{O}$. The Cauchy tensor is given by:

$$\sigma(u, p) = 2\nu\epsilon(u) - p\mathrm{Id}.$$

Let us give the equations satisfied by the motion of the structure. We take again the notations of paragraph 2.1.1: the vectors a and ω are solution of (1.6) and (1.7). In dimension 2, the rotation matrix is simply characterized by an angle $\theta \in \mathbb{R}$ and we can replace (1.7) by the equation satisfied by θ: for all $t \in (0, T)$

$$j\ddot{\theta}(t) = \int_{\partial\Omega_S(t)} (\sigma(u, p)n) \cdot (x - a(t))^{\perp} \, d\gamma \tag{1.55}$$

where $j > 0$ is the scalar moment of inertia of the structure and where we have used the following notation:

$$\forall\, x = (x_1, x_2)^T \in \mathbb{R}^2, \ x^{\perp} = (-x_2, x_1)^T.$$

In that case, the eulerian velocity of the structure is given by: $\forall\, t > 0$, $\forall\, x \in \Omega_S(t)$

$$u_S(t, x) = \dot{a}(t) + \dot{\theta}(t)(x - a(t))^{\perp}. \tag{1.56}$$

This system is completed by the boundary conditions (1.16) where u_S is given by (1.56) in dimension 2 and by (1.17) in dimension 3.

To summarize, we consider the system given by the equations (1.54), (1.55), (1.6) and (1.16) in dimension 2 and (1.54), (1.6), (1.7) and (1.16) in dimension 3. We will now state the results obtained in [BO08] et in [BG13] and give a sketch of the proofs. For a clearer presentation, the results are presented separately.

3.2 A controllability result in dimension 2 - [BO08]

Here, we consider the system of equations (1.54), (1.55), (1.6) and (1.16) completed by the initial conditions:

$$u(0,\cdot) = u_0 \text{ in } \Omega_F(0),\ a(0) = a_0,\ \dot{a}(0) = a_1,\ \theta(0) = \theta_0,\ \dot{\theta}(0) = \theta_1, \tag{1.57}$$

with $u_0 \in H^3(\Omega_F(0))$, $a_0 \in \mathbb{R}^2$ the position of the center of mass at initial time, $a_1 \in \mathbb{R}^2$, $\theta_0 \in \mathbb{R}$ and $\theta_1 \in \mathbb{R}$. We assume that these data satisfy compatibility conditions that we do not detail here. We assume that the solid satisfies some symmetry property:

$$\int_{\partial\Omega_S(0)} (y - a_0)\, d\gamma = 0. \tag{1.58}$$

This hypothesis is satisfied for a ball, an ellipse and more generally any structure symmetric with respect to its center of mass.
The statement of the controllability result is the following:

Theorem 1.8 *Let us take again the previous hypotheses and suppose that (1.53) holds. Let $T > 0$ be a fixed final time. Then, there exists $\varepsilon > 0$ depending on T and on the domains Ω, $\mathcal{O}$ and $\Omega_S(0)$ such that, if*

$$\|u_0\|_{H^3(\Omega_F(0))} + |a_0| + |a_1| + |\theta_0| + |r_0| \leqslant \varepsilon,$$

there exists a function $f \in L^2((0,T) \times \mathcal{O})$ such that the solution of (1.54), (1.55), (1.6), (1.16) and (1.57) satisfies

$$u(T,\cdot) = 0 \textit{ in } \Omega_F(T),\ a(T) = 0,\ \dot{a}(T) = 0,\ \theta(T) = 0,\ \dot{\theta}(T) = 0. \tag{1.59}$$

Remark 1.9 *This result allows to get in a direct way the null controllability if the control acts on a part of the external boundary of the fluid. To do so, it is sufficient to use a classical argument which consists of extending the fluid domain around the part where the control acts in order to come back to a control inside the fluid domain.*

The controllability of the incompressible Navier-Stokes equations is an active subject of research. Concerning the controllability to the trajectories, it has been first studied in [Ima01], then the result has been improved in [FCGIP04]. The proof of our Carleman inequality is inspired by this paper. Concerning the controllability results in fluid-structure interaction, a first result on a simplified problem in dimension 1 has been proved in [DFC05]. In their paper, the structure is represented by a point which evolves in the interval $(-1, 1)$ inside a fluid described by the viscous Burgers equations and the control is exerted at the boundary points (at 1 and -1). This result has been extended in [LTT13] to the case of a control which only acts on one end of the interval.

Let us mention that, for the compressible Navier-Stokes equation, the controllability for the 1D-case has been addressed in [EGGP12].

In an independent and simultaneous way, a similar result to the one given by Theorem 1.8 has been proved by O. Imanuvilov and T. Takahashi in [IT07]. In their paper, they assume that the solid is a ball. Their method needs less regularity on the fluid velocity (they only assume that $u_0 \in H^1(\Omega_F(0))$).

The first step of the proof of our theorem consists of showing a Carleman inequality for the adjoint problem associated to a linarized problem. Let a fluid velocity $\hat{u}$ and a structure displacement $\hat{a}$ and $\hat{\theta}$ be given. We assume that they satisfy

$$d\left(\overline{\widehat{\Omega}_S(t)}, \partial\big(\Omega \setminus \mathcal{O}\big)\right) \geqslant \eta, \, \forall \, t \in [0,T]$$

where $\eta > 0$ is a fixed small enough real number. The domain $\widehat{\Omega}_S(t)$ is defined by $\widehat{\Omega}_S(t) = \hat{\chi}_S(t, \Omega_S(0))$ where the flow $\hat{\chi}_S$ is given by: for all $(t, y) \in (0,T) \times \Omega_S(0)$

$$\hat{\chi}_S(t,y) = \hat{a}(t) + R_{\hat{\theta}(t)-\theta_0}(y - a_0).$$

We then introduce the adjoint problem satisfied by (v, π, b, τ) :

$$\begin{cases} -\partial_t v(t,x) - (\hat{u}\cdot\nabla)v(t,x) - \nabla\cdot\sigma(v,\pi)(t,x) = 0 & x \in \widehat{\Omega}_F(t), \\ \nabla \cdot v(t,x) = 0 & x \in \widehat{\Omega}_F(t), \\ v(t,x) = 0 & x \in \partial\Omega, \\ v(t,x) = \dot{b}(t) + \dot{\tau}(t)(x - \hat{a}(t))^{\perp} & x \in \partial\widehat{\Omega}_S(t), \\ m\ddot{b}(t) = -\displaystyle\int_{\partial\widehat{\Omega}_S(t)} (\sigma(v,\pi)n)(t,x)\, d\gamma, & \\ j\ddot{\tau}(t) = -\displaystyle\int_{\partial\widehat{\Omega}_S(t)} (\sigma(v,\pi)n)(t,x)\cdot(x - \hat{a}(t))^{\perp}\, d\gamma, & \\ v(T,\cdot) = v_T \ \text{ in } \widehat{\Omega}_F(T),\, b(T) = b_0^T,\, \dot{b}(T) = b_1^T,\, \tau(T) = \tau_0^T,\, \dot{\tau}(T) = \tau_1^T, & \end{cases} \tag{1.60}$$

where we have denoted $\widehat{\Omega}_F(t) = \Omega \setminus \overline{\widehat{\Omega}_S(t)}$.

The weights which are considered in the Carleman inequality depend on time since they follow the fluid domain (we could also have chosen, as for instance in [IT07], to express the problem in the initial configuration through a change of variables). Thus, we introduce a weight β_0 in $C^2(\Omega_F(0))$ defined by

$$\begin{aligned} &\beta_0 = 0 \text{ on } \partial\Omega \cup \partial\Omega_S(0),\ \beta_0 > 0 \text{ in } \Omega_F(0), \\ &\nabla\beta_0 \cdot n \leqslant c_1 < 0 \text{ on } \partial\Omega, \nabla\beta_0\cdot n \geqslant c_2 > 0 \text{ on } \partial\Omega_S(0),\ |\nabla\beta_0| > 0 \text{ in } \Omega_F(0)\setminus\overline{\mathcal{O}_0} \end{aligned}$$

where $\mathcal{O}_0 \subset\subset \mathcal{O}$ is a nonempty open set. We then define

$$\beta(t,x) = \beta_0(\hat{\chi}^{-1}(t,x)),\, \forall\, x \in \widehat{\Omega}_F(t),\, \forall\, t \in (0,T)$$

where $\hat{\chi}$ extends in a regular way the flow $\hat{\chi}_S$ to the fluid domain and keeps $\mathcal{O}$ fixed. Then the weight β satisfies:

$$\begin{aligned} &\beta = 0 \text{ on } \partial\Omega \cup \partial\widehat{\Omega}_S(t),\ \beta > 0 \text{ in } \widehat{\Omega}_F(t), \\ &\nabla\beta \cdot n \leqslant c_1 < 0 \text{ on } \partial\Omega, \nabla\beta\cdot n \geqslant c_2 > 0 \text{ on } \partial\widehat{\Omega}_S(t),\ |\nabla\beta| > 0 \text{ in } \widehat{\Omega}_F(t)\setminus\overline{\mathcal{O}_0}. \end{aligned}$$

Let us now introduce the weight functions, for all $\lambda \geqslant 1$, $\forall\, t \in (0,T)$, $\forall\, x \in \widehat{\Omega}_F(t)$

$$\alpha(t,x) := \frac{e^{(2k+2)\lambda M} - e^{\lambda(2kM+\beta(t,x))}}{t^k(T-t)^k}, \; \xi(t,x) := \frac{e^{\lambda(2kM+\beta(t,x))}}{t^k(T-t)^k}, \tag{1.61}$$

with their maximal and minimal values, at a fixed time t:

$$\hat{\alpha}(t) := \frac{e^{(2k+2)\lambda M} - e^{(2k+1)\lambda M}}{t^k(T-t)^k}, \; \alpha^*(t) := \frac{e^{(2k+2)\lambda M} - e^{2k\lambda M}}{t^k(T-t)^k},$$
$$\xi^*(t) := \frac{e^{2k\lambda M}}{t^k(T-t)^k}, \; \hat{\xi}(t) := \frac{e^{(2k+1)\lambda M}}{t^k(T-t)^k}.$$

where $M := \|\beta_0\|_{L^\infty(\Omega_F(0))}$ and $k \geqslant 4$. We have the following Carleman inequality:

Proposition 1.10 *There exists a constant C and two constants $\hat{s}$ and $\hat{\lambda}$ such that, for all $v_T \in L^2(\widehat{\Omega}_F(T))$, $b_0^T \in \mathbb{R}^2$, $b_1^T \in \mathbb{R}^2$, $\tau_0^T \in \mathbb{R}$, $\tau_1^T \in \mathbb{R}$, the corresponding solution (v,π,b,τ) of (1.60) satisfies, for all $s \geqslant \hat{s}$ and $\lambda \geqslant \hat{\lambda}$,*

$$s^3\lambda^4 \int_0^T\int_{\widehat{\Omega}_F(t)} e^{-2s\alpha}\xi^3|v|^2\,dx\,dt + s\lambda\int_0^T e^{-2s\alpha^*}\xi^*\left(\left|\ddot{b}\right|^2 + |\dot{\tau}|^2\right)\,dt$$
$$\leqslant Cs^2\lambda^{13}\int_0^T\int_{\mathcal{O}} e^{2s\alpha^*-4s\hat{\alpha}}\hat{\xi}^{10}|v|^2\,dx\,dt. \tag{1.62}$$

The proof of this inequality follows the scheme of the proof of the Carleman inequality proved in [FCGIP04] for a fluid alone:

- *Step 1:* The first step adapts the classical proof of the Carleman inequality for the heat equation presented in [FI96] to the following equation:

  $$-\partial_t v(t,x) - (\hat{u}\cdot\nabla)v(t,x) - \nu\Delta v(t,x) = -\nabla p \text{ in } \widehat{\Omega}_F(t).$$

  This equation is set on a moving domain. Moreover, the boundary terms have to be considered carefully but, due to the specific writing of the fluid velocity on the boundary of the solid, they can be estimated without major difficulty. In particular, we can notice that, on $\partial\widehat{\Omega}_S(t)$, $|\nabla v| \leqslant C(|\nabla v\, n| + |v|)$. At this stage, we get a Carleman inequality with pressure terms in the right-hand side.

- *Step 2:* In order to estimate the pressure terms, we notice that, taking the divergence of the first equation of system (1.60), we have

  $$\Delta\pi = \nabla\cdot\big((\hat{u}\cdot\nabla)v\big) \text{ in } \widehat{\Omega}_F(t). \tag{1.63}$$

  Then, we use the H^{-1} Carleman estimate shown in [IP03] for the elliptic equation satisfied by π. This estimate allows to get a Carleman inequality similar to (1.62) but with additional terms in π in the local integral of the right-hand side. At this stage, this inequality would allow to get a controllability result with two controls: in addition to the momentum equation, it would be necessary to control also the divergence of the velocity.

- *Step 3:* In the last step which is the most technical, the local terms with the pressure are replaced by the time derivative of the velocity and the second derivative with respect to space thanks to the first equation of (1.60). These terms are estimated through regularity estimates on intermediate problems.

From the Carleman inequality given by Proposition 1.10, we deduce an observability inequality which allows to obtain the controllability of the fluid velocity and the structure velocity for the linearized problem. To control the position of the structure, i.e. to get $a(T) = 0$ in $\mathbb{R}^2$ and $\theta(T) = 0$ in $\mathbb{R}$, we show that this will be satisfied if the control satisfies three constraints (one constraint for each scalar condition) which involve intermediary adjoint systems (following a method introduced by [Nak04]). We thus show a second observability inequality which takes into account these constraints and allows to show the controllability of the velocities and of the position of the structure for the linearized problem.

At last, we use a fixed point theorem to pass to the local null controllability for the nonlinear problem. The idea is to show the existence of a fixed point for the application which, to $(\hat{u}, \hat{a}, \hat{\theta})$, associates the set of the controlled solutions of the linearized problem:

$$\big\{(u, a, \theta) \text{ which satisfies } u(T) = 0,\ a(T) = 0, \dot{a}(T) = 0,\ \theta(T) = 0,\ \dot{\theta}(T) = 0 \text{ for a control } f\big\}.$$

Under this form, the problem is defined in a non standard way since the spaces of departure and of destination of this application depend on $\hat{a}$ and $\hat{\theta}$. We thus decompose the solution in fixed spaces and we use Kakutani's fixed point theorem for multi-valued applications.

Remark 1.11 *In this work, we used hypothesis (1.58) to deduce in a very simple way estimates on $\dot{a}$ and $\dot{\theta}$ from estimates of u in the L^2-norm on the solid boundary $\partial\widehat{\Omega}_S(t)$. Indeed, since $u(t, \cdot) = \dot{a}(t) + \dot{\theta}(t)(x - a(t))^{\perp}$ on $\partial\widehat{\Omega}_S(t)$, we have*

$$\int_{\partial\widehat{\Omega}_S(t)} |u|^2\, dx = |\dot{a}|^2 \int_{\partial\widehat{\Omega}_S(t)} 1\, dx + |\dot{\theta}|^2 \int_{\partial\widehat{\Omega}_S(t)} |x - a|^2\, dx,$$

thanks to hypothesis (1.58). In fact, even if we do not have this hypothesis, we can easily get the following inequality:

$$|\dot{a}|^2 + |\dot{\theta}|^2 \leqslant C\|u(t, \cdot)\|^2_{L^2(\partial\widehat{\Omega}_S(t))}.$$

This remark is used in [BG13] presented in the next subsection to consider a solid of arbitrary shape.

3.3 A controllability result in dimension 3 - [BG13]

Let us now consider the system of equations (1.54), (1.6), (1.7) and (1.16) and the initial conditions

$$u(0, \cdot) = u_0 \text{ in } \Omega_F(0),\ a(0) = a_0,\ \dot{a}(0) = a_1,\ Q(0) = Q_0,\ \omega(0) = \omega_0, \tag{1.64}$$

with $u_0 \in H^2(\Omega_F(0))$, $a_0 \in \mathbb{R}^3$, $a_1 \in \mathbb{R}^3$, $Q_0 \in SO_3(\mathbb{R})$ and $\omega_0 \in \mathbb{R}^3$. We assume that these data satisfy compatibility conditions that we do not detail here. We have the following result:

Theorem 1.12 *Let us take again the previous hypotheses and suppose that (1.53) holds. Let $T > 0$ be a fixed final time. Then, there exists $\varepsilon > 0$ depending on T and on the domains Ω, $\mathcal{O}$ and $\Omega_S(0)$ such that, if*

$$\|u_0\|_{H^2(\Omega_F(0))} + |a_0| + |a_1| + |Q_0 - Id| + |\omega_0| \leqslant \varepsilon,$$

there exists a function $f \in L^2(0, T; H^1(\mathcal{O}))$ such that the solution of (1.54), (1.6), (1.7), (1.16) and (1.64) satisfies

$$u(T, \cdot) = 0 \textit{ in } \Omega_F(T),\ a(T) = 0,\ \dot{a}(T) = 0,\ Q(T) = Id,\ \omega(T) = 0. \tag{1.65}$$

This result is proved following the same steps as previously. It relies on a Carleman inequality which is very closed to the one given by Proposition 1.10 but the proof of the inequality is rather different. By this way, the weights in the inequality are slightly different and in particular we assume that $k \geqslant 24$ in the definitions (1.61).

Let us briefly describe the new scheme of the proof of the Carleman inequality. First, if we take the curl of the first equation in (1.60), this allows to cancel the pressure and we get:

$$-\partial_t(\nabla \times v) - (\hat{u} \cdot \nabla)(\nabla \times v) - \nu\Delta(\nabla \times v) = L(\hat{u}, v) \text{ in } \widehat{\Omega}_F(t),$$

where $L(\hat{u}, v)$ is a first order term in v which satisfies

$$|L(\hat{u}, v)| \leqslant C|\nabla\hat{u}|\,|\nabla v|.$$

For this equation, we have a Carleman inequality obtained similarly as the one at Step 1 of the proof of Proposition 1.10 and we get a weighted estimate on $\nabla \times v$.
Then, to deduce an estimate on v, we notice that v is solution of:

$$\Delta v = -\nabla \times (\nabla \times v) \text{ in } \widehat{\Omega}_F(t), \tag{1.66}$$

completed by Dirichlet boundary conditions and we use the classical elliptic Carleman estimate ([FI96]). The boundary terms coming from the first step are then estimated thanks to regularity results proved on intermediate linearized problems.

This method is more concise than the method presented in the previous paragraph. In Step 2 of the previous method, we apply to system (1.63) an elliptic Carleman inequality with a right-hand side in H^{-1} proved in [IP03]. This estimate which allows us to estimate the pressure is a very tricky result to prove. Here, to get an estimate on v, we only apply to system (1.66) the elliptic Carleman inequality for a right-hand side in L^2. This inequality is a much more classical result which can be proved through direct computations.

Let us end this subsection with a remark on the controllability to the trajectories. In the works that I have presented, we want to control the system to zero by reaching the target $(u(T), a(T), \dot{a}(T), Q(T), \omega(T)) = (0, 0, 0, \mathrm{Id}, 0)$. Due to the dissipative and non-reversible properties of the system, we know that we are not able to reach any arbitrary target. However, we could be interested by the controllability to the trajectories. For instance, the paper [FCGIP04] shows the local controllability to the trajectories for the Navier-Stokes equations. More precisely, it shows the existence of a control v which allows to get a velocity u solution of (1.54) which satisfies at time T:

$$u(T, \cdot) = \bar{u}(T, \cdot)$$

where $\bar{u}$ is a solution of the Navier-Stokes equations without control:

$$\begin{cases} (\partial_t\bar{u} + (\bar{u} \cdot \nabla)\bar{u}) - \nabla \cdot \sigma(\bar{u}, \bar{p}) = 0 \\ \nabla \cdot \bar{u} = 0. \end{cases}$$

This result is valid if the initial conditions for u and $\bar{u}$ are close enough and if $\bar{u}$ is regular enough. Here, we could also be interested by the controllability to the trajectories for system (1.54), (1.6), (1.7) and (1.16) around a target trajectory $(\bar{a}, \bar{\omega}, \bar{u}, \bar{p})$. There should not be any major difficulty to get the controllability to the trajectories with the methods of [BO08] and [BG13] but the proof would be more technical. In particular, to consider the difference between the fluid velocities, it is necessary to make a change of variables since the velocities are not defined on the same domains and with this change of variables many additional terms appear.

3.4 Conclusion

To move from the controllability of the linear problem to the controllability of the nonlinear problem (through a fixed point theorem in the works presented previously or through the inverse function theorem in [FCGIP04]), we need to be around 0 and we thus get a local result. Let us mention that the paper [Cor96] gives a global controllability result for the Navier-Stokes equations with Navier conditions. More recently, [GIP12] has shown the global approximated controllability for the Navier-Stokes equations defined on a cube, with a control acting on all the sides of the cube except one. This result relies on the return method, one of the most used methods to get the global controllability for models in fluid mechanics. The return method has been introduced by J.-M. Coron in [Cor92] to study the stabilization of some control problems and then used in [Cor93] to prove the global exact controllability of Euler equations. Being able to use the return method to show a global controllability result on the fluid-structure interaction problems that have been presented in this section seems difficult. It would be interesting to start with a simplified problem (having in mind that the 1D Burgers equation is not globally controllable [GI07]).

The controllability for the coupling between an elastic structure and an incompressible fluid is a difficult problem which is still far from being completely solved. Let us quote the works [RV10] in the linear case and [Leq13] in the nonlinear case. These works deal with the interaction between an incompressible fluid and a beam which occupies a part of the boundary of the fluid domain and whose displacement is represented by an ODE system. A few controllability results have been obtained in the case of the interaction between a parabolic equation and a hyperbolic equation ([ZZ04] in dimension 1). A first problem to tackle in dimension 2 could be the interaction with a beam which occupies a part of the fluid domain in fixed domains.

The controllability results proved in this section, as well as in the quoted papers, are far from the applications and by the way they are not intuitive: acting on the structure by exerting a force on a subdomain of the fluid and leading all the system to rest does not seem natural ! The obtained results rather give an information on the mathematical properties of a model which couples a fluid and a rigid structure. In the case of fluid-structure interaction problems, the problem we are dealing with could be settled in a more practical way as, for instance, controlling the vorticity above the wing of a plane or controlling the sediment at the level of an hydraulic dam. These more applied problems are often reformulated as problems of optimization or optimal control where numerous numerical methods allow an approximated resolution (such exemple is presented in [FCHK13]).

As we will see in the following chapter, the techniques involved in the theoretical resolution of inverse problems are closely related to the controllability methods. In particular, in the perspectives presented at the end of the next chapter, we will mention the problem of detecting a solid immersed in a fluid.

Chapter 2

Parameter identification for a model of the respiratory airflow

1 Introduction

In this chapter, we are interested by inverse problems coming from the modelling of the airflow in the respiratory tract. The mathematical modelling in this domain of biomedical application has received a lot of interest in the past few years. In this introduction, I will quickly describe the physiological properties of the respiratory tract and present a simplified version of the model introduced in [BGM10] (for more details, we refer to the introduction in [Egl12]). I will then present the inverse problem that we want to address.

The respiratory tree is made up of the trachea which ramifies into the bronchi, then the bronchioles and at last the alveoli. At each division point or generation, the airway splits into two or more airways and at last we have hundred millions of alveoli. For this multi-scale geometry, a classical strategy is to derive models which give a precise description of the air flow in the upper part and a simplified description of the air flow in the lower part.

A model based on PDE will describe the evolution of the air flow in the truncated domain which corresponds to the upper part of the tree. We denote by $\Omega \subset \mathbb{R}^d$ for $d = 2, 3$ this domain. We assume that the velocity u and the pressure p satisfy the unsteady Stokes equation:

$$\left\{\begin{array}{rcll} \partial_t u - \nu \Delta u + \nabla p & = & 0, & \text{in } Q, \\ \nabla \cdot u & = & 0, & \text{in } Q, \end{array}\right. \tag{2.1}$$

where $Q = (0,T) \times \Omega$. This system is complemented with boundary conditions on the boundary of Ω which is the union of several subdomains:

$$\partial\Omega = \Gamma_0 \cup \Gamma_l \cup (\cup_{1 \leqslant i \leqslant N} \Gamma_i) \tag{2.2}$$

where Γ_0 corresponds to the inlet of the domain, Γ_l is the lateral boundary and, for $1 \leqslant i \leqslant N$, Γ_i corresponds to the artificial outlets (see Figure 2.1).

We then prescribe the following boundary conditions:

$$\left\{\begin{array}{rcll} u & = & 0, & \text{on } (0,T) \times \Gamma_l, \\ \nu \nabla u \, n - pn & = & -P_0 n, & \text{on } (0,T) \times \Gamma_0, \\ \nu \nabla u \, n - pn & = & -\pi_i n, & \text{on } (0,T) \times \Gamma_i, \text{ for } 1 \leqslant i \leqslant N, \end{array}\right. \tag{2.3}$$

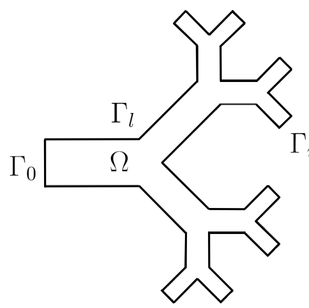


Figure 2.1: Example of geometry.

where n is the outward unit normal vector to $\partial\Omega$, P_0 is the external pressure and, for $1 \leqslant i \leqslant N$, π_i corresponds to the pressure at the artificial boundary Γ_i.

In the lower part, the flow is assumed to be laminar: by analogy with an electric network, we can consider that the flow beyond Γ_i is characterized by a unique equivalent resistance R_i. Then the flow satisfies Poiseuille's law:

$$\pi_i - P_a = R_i \left(\int_{\Gamma_i} u_i \cdot n d\gamma \right)$$

where P_a is the pressure in the acini which is assumed to be known and constant. This is a simplification in our problem: the more realistic model presented in [BGM10] includes the driving motion of the thoracic cage which is coupled with the unknown and unsteady pressure P_a.

Thus, the boundary condition on Γ_i becomes:

$$\nu \nabla u\, n - pn + R_i \left(\int_{\Gamma_i} u \cdot n\, d\gamma \right) n = -P_a n, \text{ on } (0,T) \times \Gamma_i, \text{ for } 1 \leqslant i \leqslant N. \tag{2.4}$$

The mathematical analysis of this problem has been made in [BGM10] for the Navier-Stokes system. The result relies on recent results on the regularity of Stokes problem with mixed boundary conditions in non smooth domain and gives the existence and uniqueness of solution defined locally in time.

This model presents several parameters which have to be tuned. In particular, the resistances R_i are parameters which come from the modeling formalism and it seems unrealistic to measure them directly. On the other hand, it is important to evaluate them because they vary in the case of pathology: in particular, asthma attack corresponds to an increasing of the R_i's. Thus, an interesting question is to know whether we are able to identify these parameters from measurements on the external boundary Γ_0 (i.e. at the level of the mouse). Before addressing this problem from a numerical point of view with synthetic or real measurements, a mathematical analysis allows to know if these parameters of interest are identifiable and if some stability properties hold for the identification problem. In this chapter, I will give different stability estimates related to this problem.

The boundary conditions on Γ_i are non-standard and involve non-local terms. In the studies that are presented afterwards, we have chosen to replace them with the more usual Robin boundary conditions:

$$\nu \nabla u\, n - pn + qu = -P_a n, \text{ on } (0,T) \times \Gamma,$$

where q is a variable Robin coefficient and $\Gamma = \cup_{1 \leqslant i \leqslant N} \Gamma_i$. Let us notice that, if we consider $p - P_a$ instead of p, we can replace the right-hand side by 0.

Up to our knowledge, the results that I will present are the first results which address this kind of inverse problem. The identification of a Robin coefficient for the Laplace equation or for the heat equation has been tackled in several papers like [AS06], [CFJL04] and [BCC08] (see also the references therein).

The three next sections present stability estimates of logarithmic type in different cases. The stationary Stokes problem is studied in the works presented in Section 2 which gives a result in dimension 2 and in Section 3 where the result holds in any dimension. In this second work, the stability inequalities rely on estimates which quantify the unique continuation property for the stationary Stokes problem. These estimates hold without prescribing boundary conditions on the Stokes problem. Due to their generality, they have their own interest and can be applied in other contexts.

The unsteady Stokes problem is considered in Section 4. In fact, the results presented in Sections 2 and 3 allow to get stability estimates for the unsteady Stokes problem but they hold under restrictive conditions (Robin coefficient independent of time, measurements during an infinite time). The proof of the result for the unsteady case presented in Section 4 does not rely on the steady case and gives far more general results. Again, the stability estimates rely on the quantification of the unique continuation property.

In the conclusion, I will present the difficulties which appear when we want to return to the original dissipative boundary conditions (2.4). I will also present some perspectives to these works.

2 A first stability result in dimension 2 - [BEG13b]

This section presents results obtained with Anne-Claire Egloffe and Céline Grandmont during the PhD of Anne-Claire that I co-supervised with Céline.

2.1 Context and main results

In paper [BEG13b], we consider the steady Stokes problem in dimension 2

$$\left\{ \begin{array}{rcll} -\Delta u + \nabla p & = & 0, & \text{in } \Omega, \\ \nabla \cdot u & = & 0, & \text{in } \Omega, \end{array} \right. \tag{2.5}$$

on a simplified geometry of Ω: we assume that the boundary $\partial\Omega$ is composed of only two open non-empty parts Γ_0 (where Neumann boundary conditions are prescribed) and Γ (where Robin boundary conditions are prescribed). Moreover, we assume that $\overline{\Gamma} \cap \overline{\Gamma}_0 = \emptyset$. Thanks to this assumption, we avoid problems of connection between boundary conditions of different types and we are able to get global regular solutions for our problem. In the general case (corresponding to Figure 2.1), as explained in the next section, we can work on a regular subdomain $\widetilde{\Omega} \subset \Omega$ which avoids the corner where there are changes in the boundary conditions.

Our objective is thus to identify the Robin coefficient q from measurements on u and p on Γ_0 for the problem (2.5) completed by the boundary conditions:

$$\left\{ \begin{array}{rcll} \nabla u\, n - pn & = & g, & \text{on } \Gamma_0, \\ \nabla u\, n - pn + qu & = & 0, & \text{on } \Gamma, \end{array} \right. \tag{2.6}$$

where g is non identically null. We denote by (u_i, p_i) the solution of problem (2.5)-(2.6) associated to $q = q_i$, for $i = 1, 2$.

Let us first recall the unique continuation property proved in [FL96]:

Lemma 2.1 [FL96] *Let Ω be a Lipschitz domain, let γ be a nonempty open set included in $\partial\Omega$ and $(u,p) \in H^1(\Omega) \times L^2(\Omega)$ be a solution of system (2.5) satisfying $u = 0$ and $\nabla u\, n - pn = 0$ on γ. Then $u = 0$ and $p = 0$ in Ω.*

According to this result, our parameter is identifiable in the sense that, if $u_1 = u_2$ on a nonempty open part of Γ_0, this implies that $q_1 = q_2$ on Γ.

Next, to prove stability estimates, we start with the following equality on Γ:

$$(q_2 - q_1)u_1 = q_2(u_1 - u_2) + \nabla(u_1 - u_2)n - (p_1 - p_2)n. \tag{2.7}$$

Starting with this inequality, we see that we will be able to estimate $q_2 - q_1$ on a subset where u_1 does not cancel. According to the unique continuation property, u_1 can not cancel on the whole part Γ and we have the existence of K a compact subset of Γ such that $|u_1| \geqslant m$ on K for some $m > 0$.

If we define $u = u_1 - u_2$ and $p = p_1 - p_2$, we see that, to estimate the difference $q_2 - q_1$ on K, it is sufficient to estimate

$$\|u\|_{L^2(K)} + \|\nabla u\|_{L^2(K)} + \|p\|_{L^2(K)}$$

with respect to the same kind of norm on Γ_0. In [BEG13b], we prove the following result

Lemma 2.2 *Assume that the domain Ω is of class $C^{3,1}$ and that Γ_1 and Γ_2 are nonempty parts of $\partial\Omega$. Let $(u,p) \in H^4(\Omega) \times H^3(\Omega)$ be the solution in Ω of (2.5). We assume that there exists $A > 0$ such that*

$$\|u\|_{H^4(\Omega)} + \|p\|_{H^3(\Omega)} \leqslant A. \tag{2.8}$$

Then there exist $C > 0$, which depends on A, and C_1 such that:

$$\|u\|_{L^2(\Gamma_1)} + \|\nabla u\|_{L^2(\Gamma_1)} + \|p\|_{L^2(\Gamma_1)} + \|\nabla p\|_{L^2(\Gamma_1)} \leqslant \frac{C}{\left(\log\left(\frac{C_1}{\|u\|_{L^2(\Gamma_2)} + \|\nabla u\, n\|_{L^2(\Gamma_2)} + \|p\|_{L^2(\Gamma_2)} + \|\nabla p\cdot n\|_{L^2(\Gamma_2)}}\right)\right)^{\frac{1}{2}}}. \tag{2.9}$$

Then, Lemma 2.2 and regularity results on problem (2.5)-(2.6) allow to prove the following stability result for the identification of the Robin parameter:

Theorem 2.3 *We assume that the domain Ω is of class $C^{3,1}$. Let $\alpha > 0$, $M_1 > 0$, $M_2 > 0$, $(g, q_j) \in H^{\frac{5}{2}}(\Gamma_0) \times H^{\frac{5}{2}}(\Gamma)$ for $j = 1, 2$ be such that $\|g\|_{H^{\frac{5}{2}}(\Gamma_0)} \leqslant M_1$, $q_j \geqslant \alpha$ on Γ and $\|q_j\|_{H^{\frac{5}{2}}(\Gamma)} \leqslant M_2$. Let K be a compact subset of Γ such that $|u_1| \geqslant m$ on K for some $m > 0$. Then there exist positive constants C which depends on M_1, M_2 and α and C_1 such that*

$$\|q_1 - q_2\|_{L^2(K)} \leqslant \frac{C}{m}\left(\log\left(\frac{C_1}{\|u_1 - u_2\|_{L^2(\Gamma_0)} + \|p_1 - p_2\|_{L^2(\Gamma_0)} + \|\nabla(p_1 - p_2)\cdot n\|_{L^2(\Gamma_0)}}\right)\right)^{-\frac{1}{2}}. \tag{2.10}$$

The core of the proof of Theorem 2.3 is therefore given by the result stated in Lemma 2.2 whose proof is sketched in the next subsection. This result does not involve anymore the parameter identification problem and gives a stability estimate for the classical Cauchy problem which can be formulated by the following question: *is it possible to evaluate a solution of a PDE on a part of* *a boundary from measurements on another part of the boundary ?* For the Laplace equation, the Cauchy problem has been widely studied and is known to be an ill-posed problem in the sense given by Hadamard. In particular, simple counter-examples allow to show that the best possible rate of stability is logarithmic. We refer to [ARRV09] for a general presentation on this problem.

If we compare Lemma 2.2 with the associated unique continuation property (Lemma 2.1), we see that Lemma 2.2 (and thus Theorem 2.3) is not optimal for two reasons: the regularity of the domain and of the solution (the optimal spaces would be (u,p) in $H^1(\Omega)\times L^2(\Omega)$ in a Lipschitz domain) and the measurements involved in the right-hand side of (2.9). Concerning this last point, according to the note [IY15], we can in fact replace the norms $\|\nabla u\, n\|_{L^2(\Gamma_2)} + \|p\|_{L^2(\Gamma_2)}$ in the right-hand side by the norm $\|\nabla u\, n - pn\|_{L^2(\Gamma_2)}$ and, by this way, the term $\|p_1 - p_2\|_{L^2(\Gamma_0)}$ in the right-hand side of (2.10) disappears.

Remark 2.4 *Knowing whether one can estimate $q_2 - q_1$ on the whole set Γ remains an open problem for the Stokes system. In the case of the scalar Laplace equation, if $g \geqslant 0$, [CJ99] shows that, according to the maximum principle, then the solution u of*

$$\begin{cases} \Delta u &= 0, & \text{in } \Omega, \\ \nabla u\, n &= g, & \text{on } \Gamma_0, \\ \nabla u\, n + qu &= 0, & \text{on } \Gamma, \end{cases}$$

stays positive on Γ. Such a result comes from properties specific to harmonic functions. In the general case, the paper [ADPR03] proves that it is possible to find a lower bound on the velocity obtained thanks to a doubling inequality on the boundary ([ABRV00]): more precisely, they prove by using methods of complex analytic function theory that u has at most a finite number of zeros and that their vanishing rate is bounded. They are then able to get estimates of the difference between the Robin coefficients on $\{x \in \Gamma / d(x, \partial\Omega\backslash\Gamma) \geqslant d)\}$, for any $d > 0$. Due to the methods employed, it does not seem that we can extend these results to the Stokes system. This is why we will estimate the Robin coefficient on a compact subset $K \subset \Gamma$ on which u_1 does not vanish.

Let us end this subsection with a remark on the logarithmic nature of the estimates. In the proof of Theorem 2.3, we do not take advantage of the fact that we are only trying to recover a parameter involved in the boundary condition which seems less ambitious than solving the Cauchy problem. Thus, one may hope, using another method, to get a better estimate than a logarithmic one for the Robin coefficient. However, for the Laplace equation, [CFJL04] proves that the stability for the identification of the Robin coefficient is at best logarithmic. This result has been complemented by paper [Sin07] which proves that, if the Robin coefficient is piecewise continuous, then the stability inequality becomes Lipschitz and the constant in this inequality increasess exponentially with the number of pieces. As shown in [Bou13], the fact that the inequality is Lipschitz comes from a general argument which relies on the fact that the system is identifiable and that the unknown parameter is searched in a finite-dimension space. This argument is applied in [Egl13] for the Stokes system to get Lipschitz stability. Proving that the constant increases exponentially with the number of pieces as for the Laplace equation would be an interesting perspective.

2.2 Proof of Lemma 2.2

Let us now give the main arguments for the proof of Lemma 2.2. To prove this lemma, our approach was inspired by the paper [CCL08] which considers the Laplace equation: Lemma 2.2 is obtained thanks to a Carleman inequality proved by Bukhgeim [Buk93] which is given by the following lemma

Lemma 2.5 *Let $\Psi \in C^2(\overline{\Omega})$. We have:*

$$\int_\Omega (\Delta\Psi|u|^2 + (\Delta\Psi - 1)|\nabla u|^2)e^\Psi \leqslant \int_\Omega |\Delta u|^2 e^\Psi + \int_{\partial\Omega} \frac{\partial \Psi}{\partial n}\left(|u|^2 + |\nabla u|^2 + 2\left|\frac{\partial |\nabla u|^2}{\partial \tau}\right|\right) e^\Psi$$

for all $u \in H^3(\overline{\Omega})$.

This inequality is only valid in dimension 2 and its proof relies on the relation between harmonic functions and holomorphic functions in $\mathbb{C}$.

Then, the inequality of Lemma 2.5 is applied twice: to the velocity u and to the pressure p. If we sum up these inequalities, use that $\Delta u = \nabla p$ and $\Delta p = 0$ on Ω and assume that the weight Ψ satisfies $\Delta \Psi = \lambda > 2$, then we get an inequality where the integrals in Ω are positive and we are able to deduce the following inequality which only involves integrals on the global boundary $\partial\Omega$:

$$\int_{\partial\Omega} \frac{\partial \Psi}{\partial n}\left(|u|^2 + |\nabla u|^2 + 2\left|\frac{\partial |\nabla u|^2}{\partial \tau}\right| + |p|^2 + |\nabla p|^2 + 2\left|\frac{\partial |\nabla p|^2}{\partial \tau}\right|\right) e^\Psi d\gamma \geqslant 0.$$

The next step relies on an adequate choice of the weight Ψ which allows to isolate the values of u and p and their gradients on Γ_1 and to estimate them with respect to their values on Γ_2.

3 Quantification of the unique continuation property for the stationary Stokes problem - [BEG13a]

The results presented here have been obtained in collaboration with Anne-Claire Egloffe and Céline Grandmont during the PhD of Anne-Claire.

3.1 Context and main results

In [BEG13a], we prove the same kind of result as in [BEG13b] with the main difference that the dimension d is no more restricted to 2. In this presentation, we assume that $d = 3$ but the result is more general.

The method employed here relies on the quantification of the unique continuation property for the stationary Stokes problem recalled in Lemma 2.1. Let us first state this result:

Theorem 2.6 *Assume that Ω is of class C^∞. Let $0 < \nu \leqslant \frac{1}{2}$. Let Γ be a nonempty open subset of the boundary of Ω and ω be a nonempty open set included in Ω. Then, there exists $d_0 > 0$ such that for all $\beta \in \left(0, \frac{1}{2} + \nu\right)$, for all $d > d_0$, there exists $C > 0$, such that we have*

$$\|u\|_{H^1(\Omega)} + \|p\|_{H^1(\Omega)} \leqslant C \frac{\|u\|_{H^{\frac{3}{2}+\nu}(\Omega)} + \|p\|_{H^{\frac{3}{2}+\nu}(\Omega)}}{\left(\log\left(d\frac{\|u\|_{H^{\frac{3}{2}+\nu}(\Omega)} + \|p\|_{H^{\frac{3}{2}+\nu}(\Omega)}}{\|u\|_{L^2(\Gamma)} + \|\nabla u\, n - pn\|_{L^2(\Gamma)} + \|\nabla p \cdot n\|_{L^2(\Gamma)}}\right)\right)^\beta}, \tag{2.11}$$

and

$$\|u\|_{H^1(\Omega)} + \|p\|_{H^1(\Omega)} \leqslant C \frac{\|u\|_{H^{\frac{3}{2}+\nu}(\Omega)} + \|p\|_{H^{\frac{3}{2}+\nu}(\Omega)}}{\left(\log\left(d\frac{\|u\|_{H^{\frac{3}{2}+\nu}(\Omega)} + \|p\|_{H^{\frac{3}{2}+\nu}(\Omega)}}{\|u\|_{L^2(\omega)} + \|p\|_{L^2(\omega)}}\right)\right)^\beta}, \tag{2.12}$$

for all couple $(u,p) \in H^{\frac{3}{2}+\nu}(\Omega) \times H^{\frac{3}{2}+\nu}(\Omega)$ solution of (2.5).

For similar results for the Laplace equation, let us quote [Phu03], [ARRV09], [BD10]. In particular, [ARRV09] gives an optimal stability inequality, in terms of regularity of the solution and the domains and in terms of measurements.

As in the previous section, the estimates given here are not optimal. Nevertheless, their advantages are that they are satisfied without prescribing boundary conditions on the solution and that they provide global bounds both on u and p. These two points will be crucial to solve the inverse problem of identifying a Robin coefficient because we need to estimate both u and p on the whole domain, up to the boundary. In [LUW10], the authors derive an optimal three-balls inequality for the Stokes equation which only involves the L^2-norm of the velocity u (this can be compared to the three-balls inequality given below in Remark 2.12). Note yet that if we apply their result, we get this kind of inequality

$$\|u\|_{L^2(K)} \leqslant c\|u\|^{\beta}_{L^2(\omega)}\|u\|^{1-\beta}_{L^2(\Omega)} \tag{2.13}$$

for ω a nonempty open set and K a compact set, both included in Ω. Let us notice that, with this kind of result, we can directly get estimate on ∇p through the equation. Thus p could be known up to a constant which can be fixed as soon as we have measurements on p on a subdomain or on a part of the boundary. By this way, we would get local estimates on u and p but this would not allow to estimate u and p on the whole domain Ω. As soon as we want to get global estimates (and that is our case for the inverse problem we are interested in) it seems more tricky to get rid of the measurements on the pressure.

The quantification of the unique continuation property given by Theorem 2.6 is then applied to the identification of Robin coefficient for the Stokes problem (we refer to [CCL08] and in [ADPR03] for a similar problem with Laplace equation). We consider a domain Ω where the boundary of Ω is described by (2.2) and the following problem:

$$\left\{\begin{array}{rcll} -\Delta u + \nabla p & = & 0, & \text{in } \Omega, \\ \nabla\cdot u & = & 0, & \text{in } \Omega, \\ \nabla u\, n - pn & = & g, & \text{on } \Gamma_0, \\ u & = & 0 & \text{on } \Gamma_l, \\ \nabla u\, n - pn + qu & = & 0, & \text{on } \Gamma, \end{array}\right. \tag{2.14}$$

where g is not identically null.

We then have the following stability result for the identification of the Robin parameter:

Theorem 2.7 *Let $\Gamma_1 \subseteq \Gamma_0$ be a nonempty open subset of Γ_0. We assume that Γ and Γ_1 are of class C^∞. Let $\alpha > 0$, $M_1 > 0$, $M_2 > 0$. We assume that $(g, q_j) \in H^{\frac{3}{2}}(\Gamma_0) \times H^{\frac{3}{2}}(\Gamma)$, for $j = 1, 2$, are such that $\|g\|_{H^{\frac{3}{2}}(\Gamma_0)} \leqslant M_1$, $q_j \geqslant \alpha$ on Γ and $\|q_j\|_{H^{\frac{3}{2}}(\Gamma)} \leqslant M_2$. Let $K \subset\subset \Gamma$ be a compact subset of Γ such that $|u_1| \geqslant m$ on K for some $m > 0$.*

Then, for all $\beta \in (0, 1)$, there exists $C > 0$ and $C_1 > 0$ depending on α, M_1 and M_2 such that

$$\|q_1 - q_2\|_{L^2(K)} \leqslant \frac{1}{m}\frac{C}{\left(\log\left(\frac{C_1}{\|u_1-u_2\|_{L^2(\Gamma_1)}+\|\nabla(p_1-p_2)\cdot n\|_{L^2(\Gamma_1)}}\right)\right)^{\frac{3}{4}\beta}}, \tag{2.15}$$

for (u_j, p_j) solution of (2.14) with $q = q_j$, $j = 1, 2$.

To prove this result, we first introduce a subdomain $\widetilde{\Omega}$ of Ω of class C^∞ such that $\Gamma_1 \subset \partial\widetilde{\Omega}$ and $K \subset \partial\widetilde{\Omega}$. In this domain $\widetilde{\Omega}$, since we avoid the corners where boundary conditions change, we can prove that the solution (u,p) is regular enough. Starting again with (2.7), we see that

$$\|q_1 - q_2\|_{L^2(K)} \leqslant \frac{C(M_2)}{m}(\|u\|_{L^2(K)} + \|\nabla u\|_{L^2(K)} + \|p\|_{L^2(K)}) \leqslant \frac{C(M_2)}{m}(\|u\|_{H^{3/2+\epsilon}(\widetilde{\Omega})} + \|p\|_{H^1(\widetilde{\Omega})})$$

for all $\epsilon > 0$. The hypotheses on the data g, q_j, for $j = 1, 2$ allow to prove that (u,p) belongs to $H^3(\Omega) \times H^2(\Omega)$ and stays bounded in this space and thus, by interpolation, we get

$$\|q_1 - q_2\|_{L^2(K)} \leqslant \frac{C(\alpha, M_1, M_2)}{m}(\|u\|_{H^1(\widetilde{\Omega})} + \|p\|_{H^1(\widetilde{\Omega})})^\theta$$

for all $\theta < 3/4$. Then, we see that if we apply inequality (2.11) of Theorem 2.6 on $\widetilde{\Omega}$, we get Theorem 2.7. The main argument is thus contained in the quantification of the unique continuation property given by Theorem 2.6.

Let us notice that compared to Theorem 2.3, we need less regularity on the data ($H^{3/2}$ instead of $H^{5/2}$). On the other hand, the domain has to be locally very regular. Another difference lies in the power which appears in the right-hand side (we get $3\beta/4$ instead of $1/2$). Moreover, due to the interpolation argument, this power may be improved if we assume that the solution is more regular.

3.2 Proof of Theorem 2.6

The proof of Theorem 2.6 is made through the proof of intermediate results which give local estimates inside the domain or near the boundary. These results are summarized in the three following propositions.

The first proposition allows to transmit information from an open set to any relatively compact open set in Ω.

Proposition 2.8 *Let ω be a nonempty open set included in Ω and let $\hat{\omega}$ be a relatively compact open set in Ω. Then, there exist $c, s > 0$ such that for all $\epsilon > 0$, for all $(u,p) \in H^1(\Omega) \times H^1(\Omega)$ solution of (2.5),*

$$\|u\|_{H^1(\hat{\omega})} + \|p\|_{H^1(\hat{\omega})} \leqslant \frac{c}{\epsilon}\left(\|u\|_{H^1(\omega)} + \|p\|_{H^1(\omega)}\right) + \epsilon^s\left(\|u\|_{H^1(\Omega)} + \|p\|_{H^1(\Omega)}\right).$$

The second proposition allows to transmit information from any open set in Ω to some neighborhood of the boundary.

Proposition 2.9 *Assume that Ω is of class C^∞. Let $0 < \nu \leqslant \frac{1}{2}$, $x_0 \in \partial\Omega$ and let ω be an open set in Ω. There exists a neighborhood $\hat{\omega}$ of x_0 such that for all $\beta \in \left(0, \frac{1}{2} + \nu\right)$, there exists $c > 0$ such that for all $\epsilon > 0$, for all $(u,p) \in H^{\frac{3}{2}+\nu}(\Omega) \times H^{\frac{3}{2}+\nu}(\Omega)$ solution of (2.5),*

$$\|u\|_{H^1(\hat{\omega}\cap\Omega)} + \|p\|_{H^1(\hat{\omega}\cap\Omega)} \leqslant e^{\frac{c}{\epsilon}}(\|u\|_{H^1(\omega)} + \|p\|_{H^1(\omega)}) + \epsilon^\beta(\|u\|_{H^{\frac{3}{2}+\nu}(\Omega)} + \|p\|_{H^{\frac{3}{2}+\nu}(\Omega)}).$$

Finally, the third proposition allows to transmit information from a part of the boundary of Ω to a relatively compact open set in Ω.

Proposition 2.10 *Assume that Ω is of class C^∞. Let $0 < \nu \leqslant \frac{1}{2}$, Γ be a nonempty open subset of the boundary of Ω and $\hat{\omega}$ be a relatively compact open set in Ω. Then, there exists $c, s > 0$ such that for all $\epsilon > 0$, for all $(u,p) \in H^{\frac{3}{2}+\nu}(\Omega) \times H^{\frac{3}{2}+\nu}(\Omega)$ solution of (2.5),*

$$\|u\|_{H^1(\hat{\omega})} + \|p\|_{H^1(\hat{\omega})} \leqslant \frac{c}{\epsilon}\left(\|u\|_{H^1(\Gamma)} + \|p\|_{H^1(\Gamma)} + \|\nabla u\, n\|_{L^2(\Gamma)} + \|\nabla p \cdot n\|_{L^2(\Gamma)}\right) + \epsilon^s(\|u\|_{H^1(\Omega)} + \|p\|_{H^1(\Omega)}).$$

Let us notice that the logarithmic nature of the stability inequalities comes from Proposition 2.9 where an exponential appears in front of the first term of the right-hand side. If we only want to estimate u and p in the interior of Ω, it is sufficient to apply Propositions 2.8 and 2.10 and we get Hölder estimates instead of logarithmic estimates.

The method to prove these intermediate results consists in applying local Carleman estimates. We use two kinds of local Carleman inequalities: inside Ω, a Carleman inequality stated in [Hör85] is used in the proof of Proposition 2.8 and, near the boundary, a Carleman inequality due to [LR95] is used in the proofs of Propositions 2.9 and 2.10. For each proposition, these Carleman estimates are applied simultaneously to u and p and, using the fact that $\Delta u = \nabla p$ and $\Delta p = \operatorname{div}(\Delta u) = 0$ in Ω, we are then able to evaluate the right-hand sides of the inequalities.

Remark 2.11 *The local Carleman inequalities are obtained thanks to Gårding inequalities involving pseudodifferential computation. In particular, the Carleman inequality stated in [LR95] holds for elliptic equations with C^∞ coefficients set in a half-plane and to apply this inequality, we use a local straightening of the boundary and the Stokes operator. This explains the fact that, in Propositions 2.9 and 2.10 (and thus in Theorem 2.6), the boundary has to be locally C^∞. In the next section, the use of global Carleman inequality will allow to consider less regular domains.*

Remark 2.12 *The arguments of the proof of Proposition 2.8 together with Caccioppoli inequality applied to p allow to prove the following three-balls inequality: For $\delta > 0$ and $q \in \mathbb{R}^3$, there exist $c > 0$, $0 < \alpha < 1$ such that for any $(u,p) \in H^1(B(q,5\delta)) \times L^2(B(q,5\delta))$ solution of (2.5) in $B(q,5\delta)$, we have*

$$\|u\|_{H^1(B(q,2\delta))} + \|p\|_{L^2(B(q,2\delta))} \leqslant c\left(\|u\|_{H^1(B(q,\delta))} + \|p\|_{L^2(B(q,\delta))}\right)^{\alpha}\left(\|u\|_{H^1(B(q,5\delta))} + \|p\|_{L^2(B(q,5\delta))}\right)^{1-\alpha}.$$

As explained in details in [ARRV09], Carleman inequalities and three-balls inequalities are strictly intertwined and are the main tools to derive stability inequalities for the unique continuation properties.

4 Quantification of the unique continuation property for the unsteady Stokes problem - [Bou]

We now want to address the unsteady problem (2.1) to be closest to the model of the respiratory airflow presented in the introduction. As in the previous section, the inverse problem of identification of a Robin parameter is tackled through the stability of the unique continuation property. For unique continuation estimates on parabolic equations like heat equations, we refer to the surveys [Yam09] and [Ves08].

In [BEG13b] and [BEG13a], we proved that our stability estimates for the steady Stokes system can be used to get stability estimates for the unsteady problem. However these inequalities hold under the very specific assumptions that the Robin coefficient does not depend on time and that the measurements are made during an infinite time. As in [BCC08] in the case of the Laplace equation, this result relies on an estimation of the difference between the solution of the stationary problem and the solution of the non stationary problem by a function which tends to zero as t tends to infinity thanks to an inequality coming from semigroup theory. The fact to go through the steady problem to address the unsteady problem leads to restrictive assumptions. In [Bou], I have obtained more general results with a proof which tackles the unsteady problem directly.

The unique continuation stability inequalities are stated in this result:

Theorem 2.13 *We assume that $\Omega \subset \mathbb{R}^3$ is of class C^2.*

1. *Let $\Gamma \subset \partial\Omega$ be a nonempty open subset of $\partial\Omega$. There exist a constant $\alpha > 0$ and, for all $\varepsilon > 0$, there exists a constant $C > 0$ such that*

$$\|u\|_{C([\varepsilon,T-\varepsilon];C^1(\overline{\Omega}))} + \|p\|_{C([\varepsilon,T-\varepsilon]\times\overline{\Omega})} \leqslant \frac{CM}{\left(\log\left(\dfrac{CM}{G}\right)\right)^{\alpha}} \tag{2.16}$$

for all (u,p) solution of (2.1) in $\left(H^1(0,T;H^3(\Omega)) \cap H^2(0,T;H^1(\Omega))\right) \times H^1(0,T;H^2(\Omega))$. In this inequality, M is defined by

$$M := \|u\|_{H^1(0,T;H^3(\Omega))} + \|u\|_{H^2(0,T;H^1(\Omega))} + \|p\|_{H^1(0,T;H^2(\Omega))} \tag{2.17}$$

and G is defined by

$$G = \|u\|_{L^2((0,T)\times\Gamma)} + \|\nabla u\, n - pn\|_{L^2((0,T)\times\Gamma)} + \|\nabla p \cdot n\|_{L^2((0,T)\times\Gamma)}. \tag{2.18}$$

2. *Let $\hat{\omega}$ be an open subset of Ω relatively compact in Ω. There exist a constant $\alpha > 0$ and, for all $\varepsilon > 0$, there exists a constant $C > 0$ such that we have*

$$\|u\|_{C([\varepsilon,T-\varepsilon];C^1(\overline{\Omega}))} + \|p\|_{C([\varepsilon,T-\varepsilon]\times\overline{\Omega})} \leqslant \frac{CM}{\left(\log\left(\dfrac{CM}{\|u\|_{L^2((0,T)\times\hat{\omega})} + \|p\|_{L^2((0,T)\times\hat{\omega})}}\right)\right)^{\alpha}} \tag{2.19}$$

for all (u,p) solution of (2.1) in $\left(H^1(0,T;H^3(\Omega)) \cap H^2(0,T;H^1(\Omega))\right) \times H^1(0,T;H^2(\Omega))$. Here, M is again given by (2.17).

To prove this result, as for Theorem 2.6, the Stokes problem is rewritten as the coupling of a heat equation satisfied by u and the Laplace equation satisfied by p. Then, the main part of the work consists of obtaining local estimates on u and p inside the domain and near the boundary. These estimates are not given here and correspond to the counterpart in the unsteady case of Propositions 2.8, 2.9 and 2.10. They are proved thanks to local Carleman estimates for the heat equation and the Laplace equation. The method is inspired by the one presented in paper [Yam09] which considers parabolic equations like heat equation.

Contrary to the local Carleman estimates used in [BEG13a], here the local Carleman estimates are derived through direct computations. They are obtained in the same way as global Carleman inequalities thanks to the method of Fursikov and Imanuvilov [FI96]. We call them local Carleman estimates because they are stated on a subdomain of $(0,T)\times\Omega$ where we do not prescribe boundary conditions on the solutions. For the parabolic equation, the Carleman inequality that is used is stated in [Yam09] and the Carleman inequality in the elliptic case can be proved with the methods presented in [FI96]. This approach has two advantages: first, these Carlemenan estimates are obtained in a more basic and flexible way and second the domain has to be less regular. Compared to Theorem 2.6, we only need Ω to be of class C^2 which corresponds to the classical assumptions to construct the weights in the global Carleman inequalities.

The estimates in Theorem 2.6 involve the norms of u, p and their gradients in the space of continuous functions. In fact, the local estimates inside the domain can be proved in $L^2(H^1)$ but, for the estimates on the boundary, we prove pointwise estimates on u, p and their gradients. This is due to the method where we approach boundary points by a set of parabola located inside the domain in order to evaluate the values of the unknowns at these boundary points. To give sense to the norms of u and p in (2.16) and (2.19), it is natural to assume that (u,p) belongs to $H^1(0,T;H^3(\Omega))\times H^1(0,T;H^2(\Omega))$. If we do not assume that u belongs to $H^2(0,T;H^1(\Omega))$, the same kind of estimate still holds with the additional norms $\|u\|_{H^1(0,T;L^2(\Gamma))}$ in (2.18) and $\|u\|_{H^1(0,T;L^2(\hat{\omega}))}$ in (2.19). As proved in [Yam09], it is necessary to have a control with this kind of norm involving a derivative in time, otherwise the estimate fails.

Remark 2.14 *Instead of taking the divergence of the Stokes equation, another method could have been to take the curl and to prove a global estimate on the velocity alone with the help of adapted Carleman estimates (like in [CIPY13] or [BG13]). According to Stokes equation, this would allow to directly get an estimate on ∇p and, thanks to an adapted Poincaré inequality, this leads to an estimate on p if we also have measurements of p on an arbitrary sub-domain. This alternative method seems to lead to similar measurements as the ones in the inequalities given by Theorem 2.13.*

Let us mention, that, in [IK13], the authors prove a local stability estimate which only involves the velocity for the solution of the Navier-Stokes equation. They assume that the data belong to a Gevrey class and enforce specific conditions on the solution which are satisfied if periodic boundary conditions are prescribed.

In a similar way as in the previous section, we can apply this result to the inverse problem of identifying q in the following system:

$$\begin{cases} \partial_t u - \Delta u + \nabla p &= 0, & \text{in } Q,\\ \nabla\cdot u &= 0, & \text{in } Q,\\ \nabla u\, n - pn &= g, & \text{on } (0,T)\times\Gamma_0,\\ u &= 0, & \text{on } (0,T)\times\Gamma_l,\\ \nabla u\, n - pn + qu &= 0, & \text{on } (0,T)\times\Gamma,\\ u(0,\cdot) &= u_0, & \text{in } \Omega. \end{cases} \tag{2.20}$$

We then get

Theorem 2.15 *Let Ω be of class $C^{2,1}$and $\Gamma_1 \subseteq \Gamma_0$ be a nonempty open subset of the boundary of Ω. Let $\nu_0 > 0$ and $N_0 > 0$.*
Let $u_0 \in H^4(\Omega)\cap V$, $g \in H^2(0,T;L^2(\Gamma_0))\cap H^1(0,T;H^{\frac{3}{2}}(\Gamma_0))$ be non identically zero and q_1, $q_2 \in H^2(0,T;H^2(\Gamma))$ such that q_1, $q_2 \geqslant \nu_0$ on $(0,T)\times\Gamma$. We assume that

$$\|u_0\|_{H^4(\Omega)} + \|g\|_{H^2(0,T;L^2(\Gamma_0))\cap H^1(0,T;H^{\frac{3}{2}}(\Gamma_0))} + \sum_{j=1}^{2}\|q_j\|_{H^2(0,T;H^2(\Gamma))} \leqslant N_0.$$

We denote by (u_j,p_j) the solution of system (2.20) with $q = q_j$ for $j = 1,2$. Let K be a compact subset of $\{(t,x)\in(\varepsilon,T-\varepsilon)\times\Gamma \,/\, u_1 \neq 0\}$ for some $\varepsilon > 0$ and $m > 0$ be such that $|u_1| \geqslant m$ on K.

Then, there exists $\alpha > 0$ independent of ε, $C > 0$ which depends on ε, ν_0 and N_0 such that

$$\|q_1 - q_2\|_{C(K)} \leqslant \frac{1}{m} \frac{C}{\left(\log\left(\dfrac{C}{\|u_1 - u_2\|_{L^2((0,T)\times\Gamma_1)} + \|\nabla(p_1 - p_2)\cdot n\|_{L^2((0,T)\times\Gamma_1)}}\right)\right)^{\alpha}}.$$

Remark 2.16 *Many problems on identification of parameters for unsteady PDEs are solved thanks to a method due to Bukhgeim and Klibanov [BK81]. We refer to [Kli13] for a large number of applications to this method. In our case, since the coefficient appears in the boundary conditions, it is not possible to follow this approach directly.*

5 Conclusion

In all these studies, we have considered Robin boundary conditions. Let us now try to come back to the boundary conditions (2.4) presented in the introduction.

First, to have the identifiability of the coefficients $(R_i)_{1\leqslant i\leqslant N}$, it is necessary to add a condition on the flux of the solution. More precisely, if (u_1, p_1) and (u_2, p_2) are solution respectively associated to $(R_i^1)_{1\leqslant i\leqslant N}$ and $(R_i^2)_{1\leqslant i\leqslant N}$, then we have to assume that, for all $1 \leqslant i \leqslant N$, there exists $t_0 \in (0, T)$ such that

$$\int_{\Gamma_i} u_1(t_0, \cdot)\cdot n \, d\gamma \neq 0.$$

Next, if we want to get stability estimates, we write that

$$(R_i^2 - R_i^1)\left(\int_{\Gamma_i} u_1 \cdot n \, d\gamma\right) n = R_i^2 \left(\int_{\Gamma_i} (u_1 - u_2)\cdot n \, d\gamma\right) n + \nabla(u_1 - u_2)\, n - (p_1 - p_2)n.$$

Thus, due to the presence of the first term in the right-hand side, it is necessary to have an estimate on $u_1 - u_2$ on the whole part Γ_i and, compared to the classical Robin conditions, we can no more consider the solution on a subdomain far from the corners. By this way, due to the mixed boundary conditions, we do not have enough regularity on the solution to extend directly the previous stability estimates to this case.

The paper [Ves08] considers the heat equation set on a moving domain and, with similar tools as the one presented in this chapter, addresses the problem of identifying an internal moving boundary from measurements of the solution on a fixed accessible part of the boundary. It would be interesting to study a similar problem for the unsteady Stokes system and then to try to extend these methods to the interaction between a fluid modeled by the Stokes system and a rigid structure.

Chapter 3

Mathematical and numerical study of the cardiac electrical activity

1 Introduction

Describing the activity of the heart is one of the major concerns among the numerous research subjects in the mathematics applied to the biomedical field. In the functioning of the heart, it is recognized that the electrical front which initiates the contraction of the heart plays a central role. In particular, most of the heart problems come from a dysfunction of its electrical activity.

In this chapter, I will present several contributions to the study of the electrical cardiac activity at different stages (theoretical analysis, numerical simulations, inverse problems). Section 2 is a brief introduction to the modeling of the electrical activity of the heart and ends with a result on the mathematical analysis of the model. In Section 3, I present numerical simulations of electrocardiograms, the main tool to measure the electrical functioning of the heart. Then, Section 4 gives theoretical and numerical results for the identification of some model parameters. At last, Section 5 is devoted to a study on the influence of noise on the electrical behavior of a cardiac tissue. Each section ends with a presentation of ongoing works and perspectives.

2 Modeling and mathematical analysis

2.1 Modeling of the electrical activity of the heart

Let us first briefly present the model based on PDE which is the most commonly used, the bidomain model. We refer to the books [SLC+06], [PBC05], [Sac04], [CFPCS06] and to the review [CP08] for a detailed presentation. This macroscopic model which has been introduced in [Tun78] is based on the hypothesis that the cardiac tissue can be viewed as the union of two conductive media: the intracellular medium composed by cardiac cells and the extracellular medium which corresponds to the space between the cells. After a homogenization step formally presented in [NK93] and studied in a rigourous way in [PSCF05]), one can consider that the intra- and extracellular media occupy the whole cardiac domain Ω_{H}. This allows to give a sense on Ω_{H} to the variables of the problem: the intra- and extracellular current densities $\boldsymbol{j}_{\mathrm{i}}$ and $\boldsymbol{j}_{\mathrm{e}}$, the conductivity tensors $\boldsymbol{\sigma}_{\mathrm{i}}$ and $\boldsymbol{\sigma}_{\mathrm{e}}$ and the electrical potentials u_{i} and u_{e}. A first equation comes from the conservation of the current between the intra- and extra-cellular domains:

$$\operatorname{div}(\boldsymbol{j}_{\mathrm{i}} + \boldsymbol{j}_{\mathrm{e}}) = 0, \quad \text{in} \quad (0, T) \times \Omega_{\mathrm{H}}. \tag{3.1}$$

The second equation comes from the electrical balance on the cell membrane which is modeled as an electrical circuit with a resistor and capacitor in parallel:

$$A_{\mathrm{m}}\left(C_{\mathrm{m}}\partial_t V_{\mathrm{m}} + i_{\mathrm{ion}}(V_{\mathrm{m}}, w)\right) + \mathrm{div}(\boldsymbol{j}_{\mathrm{i}}) = A_{\mathrm{m}} I_{\mathrm{app}}, \quad \text{in} \quad (0,T)\times\Omega_{\mathrm{H}}, \tag{3.2}$$

completed by Ohm's laws

$$\boldsymbol{j}_{\mathrm{i}} = -\boldsymbol{\sigma}_{\mathrm{i}}\nabla u_{\mathrm{i}}, \quad \boldsymbol{j}_{\mathrm{e}} = -\boldsymbol{\sigma}_{\mathrm{e}}\nabla u_{\mathrm{e}}. \tag{3.3}$$

In equation (3.2), V_{m} is the transmembrane potential given by:

$$V_{\mathrm{m}} \stackrel{\mathrm{def}}{=} u_{\mathrm{i}} - u_{\mathrm{e}}, \tag{3.4}$$

A_{m} is a constant which represents the ratio of the cell membrane surface by unit of volume and C_{m} is the membrane capacity by unit of surface. The term $i_{\mathrm{ion}}(V_{\mathrm{m}}, w)$ represents the membrane ionic current and I_{app} is the current of the external stimulation. This current is a function of V_{m} and of an auxiliary variable w called ionic variable.

In general, the ionic variable w satisfies an ODE system of the type:

$$\partial_t w + g(V_{\mathrm{m}}, w) = 0, \quad \text{in} \quad (0,T)\times\Omega_{\mathrm{H}}. \tag{3.5}$$

The definitions of the functions g and i_{ion} depend on the considered model (see [SLC+06, PBC05] and the references therein). Since the founding works of Hodgkin and Huxley [HH52], numerous ionic models for excitable cells have been proposed in the litterature. They are classified in two categories (see [PBC05, Chapitre 3]): the phenomenological models and the physiological models. In the first case, the models try to reproduce the properties of the action potential (like its shape or the restitution curve) without relying on physiological observations. The ionic variable is often a scalar variable (FitzHugh-Nagumo [Fit61, NAY62], Roger-McCulloch [RM94], Aliev-Panfilov [AP96], Mitchell-Schaeffer [MS03]) or a vector of dimension 2 or 3 (Fenton-Karma [FK98], Bueno Orovio-Cherry-Fenton [BOCF08]). Besides, the physiological models rely on experimental observations and try to represent the exchanges which occur at the level of the cell membrane or inside the cell between the different ions. In this class, among the numerous existing models, let us quote the following models: di Francesco-Noble [dFN85], Beeler-Reuter [BR77], Luo-Rudy I [LR91], Luo-Rudy II [LR94], Courtemanche-Ramirez-Nattel [CRN98], TNNP [TTNNP04].

To summarize, the system of equations which model the electrical activity of the heart is a system which couples an elliptic equation (3.1), a nonlinear reaction-diffusion equation (3.2) and an ODE (3.5):

$$\begin{cases} -\,\mathrm{div}\left((\boldsymbol{\sigma}_{\mathrm{i}}+\boldsymbol{\sigma}_{\mathrm{e}})\nabla u_{\mathrm{e}}\right) - \mathrm{div}(\boldsymbol{\sigma}_{\mathrm{i}}\nabla V_{\mathrm{m}}) = 0 & \text{in} \quad (0,T)\times\Omega_{\mathrm{H}}, \\ \chi_{\mathrm{m}}\partial_t V_{\mathrm{m}} + I_{\mathrm{ion}}(V_{\mathrm{m}}, w) - \mathrm{div}(\boldsymbol{\sigma}_{\mathrm{i}}\nabla V_{\mathrm{m}}) - \mathrm{div}(\boldsymbol{\sigma}_{\mathrm{i}}\nabla u_{\mathrm{e}}) = I_{\mathrm{app}} & \text{in} \quad (0,T)\times\Omega_{\mathrm{H}}, \\ \partial_t w + g(V_{\mathrm{m}}, w) = 0 & \text{in} \quad (0,T)\times\Omega_{\mathrm{H}}, \end{cases} \tag{3.6}$$

completed by boundary conditions

$$\begin{cases} \boldsymbol{\sigma}_{\mathrm{i}}\nabla V_{\mathrm{m}}\cdot \mathrm{n} + \boldsymbol{\sigma}_{\mathrm{i}}\nabla u_{\mathrm{e}}\cdot \mathrm{n} = 0 & \text{on} \quad (0,T)\times\Sigma, \\ \boldsymbol{\sigma}_{\mathrm{e}}\nabla u_{\mathrm{e}}\cdot \mathrm{n} = -J_T\cdot \mathrm{n} & \text{on} \quad (0,T)\times\Sigma. \end{cases} \tag{3.7}$$

Here, $\chi_{\mathrm{m}} \stackrel{\mathrm{def}}{=} A_{\mathrm{m}}C_{\mathrm{m}}$, $I_{\mathrm{ion}} \stackrel{\mathrm{def}}{=} A_{\mathrm{m}} i_{\mathrm{ion}}$, the vector n is the normal unit vector external to $\Sigma \stackrel{\mathrm{def}}{=} \partial\Omega_{\mathrm{H}}$ and J_T corresponds to the volume current density in the torso. At last, we enforce the initial conditions $V_{\mathrm{m}}|_{t=0} = V_{\mathrm{m}}^0$ and $w|_{t=0} = w^0$.

The boundary conditions (3.7) are commonly accepted (see for instance [Tun78, PBC05, SLC$^+$06]). They express the fact that the intracellular current does not propagate outside the heart and that the current flow is continuous between the extracellular medium and the external medium. Since the current flow in the external medium is unknown, it is frequently assumed that the heart is electrically insulated. This allows to replace the previous conditions by :

$$\begin{cases} \boldsymbol{\sigma}_{\mathrm{i}} \nabla V_{\mathrm{m}} \cdot \mathrm{n} + \boldsymbol{\sigma}_{\mathrm{i}} \nabla u_{\mathrm{e}} \cdot \mathrm{n} = 0 & \text{on} \quad (0,T) \times \Sigma, \\ \boldsymbol{\sigma}_{\mathrm{e}} \nabla u_{\mathrm{e}} \cdot \mathrm{n} = 0 & \text{on} \quad (0,T) \times \Sigma, \end{cases} \tag{3.8}$$

The system of equations (3.6),(3.8) is called the *isolated bidomain* model.

2.2 Modeling of the electrocardiogram (ECG)

The electrocardiogram (ECG) is the most common clinical exam to observe the functioning of the heart and to set up a first diagnosis. It relies on the fact that the heart generates an electrical field which can be observed at the surface of the skin. The ECG is composed of 12 graphs which are usually called leads and correspond to the evolution during time of potentials or differences of potentials measured by electrodes put on the skin according to a standardized plan (see Figure 3.1).

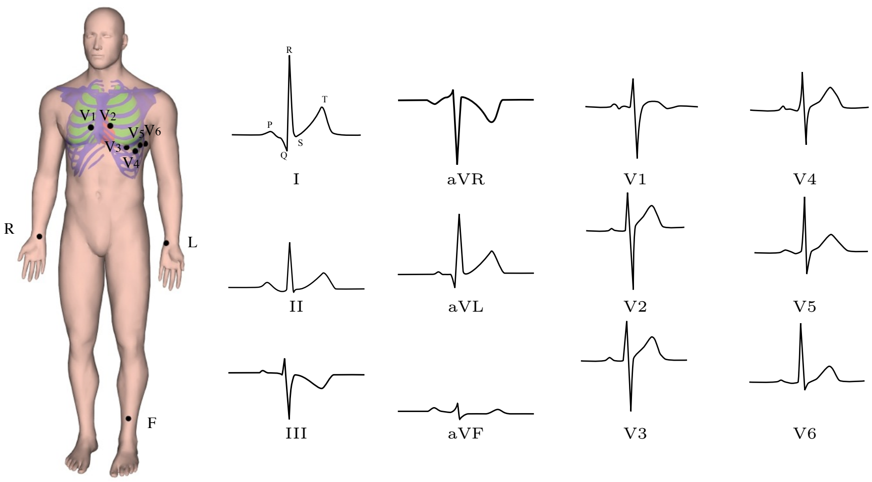


Figure 3.1: *Left*: ECG electrodes location (black dots). *Right*: reproduction of a normal 12-lead ECG: standard leads (I, II, III), augmented leads (aVR, aVL, aVF) and chest leads (V1, V2, ..., V6). For example, lead I corresponds to the difference of potential between L and R, lead II the difference between F and R, leads III the difference between F and L (the other definitions can be found, *e.g.*, in [BCF$^+$10]).

As explained for instance in [MP95], each ECG deflection corresponds to a specific electric state of the heart: the first deflection, called the P-wave (see lead I in Figure 3.1, right), corresponds to the depolarization of the atria; the group made of the second, third and fourth deflections, called the QRS-complex, corresponds to the depolarization of the ventricles; the last deflection, called the T-wave, corresponds to the repolarization of the ventricles.

Let us denote by Ω_{T} the area of the human body located outside the heart (see Figure 3.2). For convenience, we will call this region the *torso*. We assume that the torso is a passive conductor and the torso potential u_{T} is thus modeled by the generalized Laplace equations:

$$\begin{cases} -\operatorname{div}(\boldsymbol{\sigma}_{\mathrm{T}}\nabla u_{\mathrm{T}}) = 0 & \text{in} \quad (0,T)\times\Omega_{\mathrm{T}} \\ \boldsymbol{\sigma}_{\mathrm{T}}\nabla u_{\mathrm{T}}\cdot \mathrm{n} = 0 & \text{on} \quad (0,T)\times\Gamma_{\mathrm{ext}} \end{cases} \tag{3.9}$$

where $\boldsymbol{\sigma}_{\mathrm{T}}$ is the conductivity tensor in the torso. The boundary condition on $\Gamma_{\mathrm{ext}} \stackrel{\mathrm{def}}{=} \partial\Omega_{\mathrm{T}} \setminus \Sigma$ expresses the fact that no current goes through the external surface of the torso.

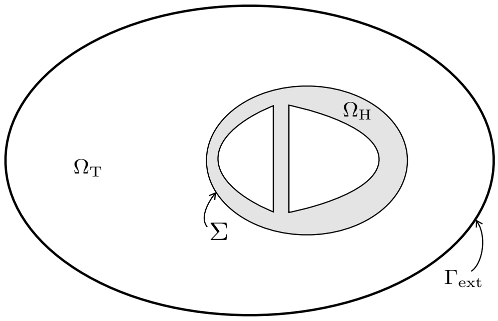


Figure 3.2: Geometrical description in dimension 2: heart domain Ω_{H}, torso domain Ω_{T}, heart-torso interface Σ and external boundary Γ_{ext}.

For the boundary conditions at the heart-torso interface Σ, in general we consider ([Tun78, KN94, PBC05, SLC+06]) that we have a perfect coupling given by the conditions:

$$\begin{cases} u_{\mathrm{e}} = u_{\mathrm{T}} & \text{on} \quad (0,T)\times\Sigma \\ \boldsymbol{\sigma}_{\mathrm{e}}\nabla u_{\mathrm{e}}\cdot \mathrm{n} = \boldsymbol{\sigma}_{\mathrm{T}}\nabla u_{\mathrm{T}}\cdot \mathrm{n} & \text{on} \quad (0,T)\times\Sigma. \end{cases} \tag{3.10}$$

The second condition corresponds to the condition $(3.7)_2$ with $J_T = -\boldsymbol{\sigma}_{\mathrm{T}}\nabla u_{\mathrm{T}}$. If we make the simplifying hypothesis that the heart is electrically isolated (condition $(3.8)_2$), this amounts to neglect the influence of the torso on the heart and the strong coupling (3.10) is thus replaced by a weak coupling:

$$\begin{cases} u_{\mathrm{T}} = u_{\mathrm{e}} & \text{on} \quad (0,T)\times\Sigma \\ \boldsymbol{\sigma}_{\mathrm{e}}\nabla u_{\mathrm{e}}\cdot \mathrm{n} = 0 & \text{on} \quad (0,T)\times\Sigma. \end{cases} \tag{3.11}$$

We will see in Section 3 the influence of the choice of the boundary conditions on the ECG obtained through numerical simulations. In [BCF+10], other types of coupling have been considered in order to take into account the possible capacitor and resistor effects of the pericardium.

2.3 Mathematical analysis - [BFGZ08]

If we replace the first equation of (3.6) by the difference between the first two equations, we see that this system may be viewed as a degenerate in time system of two reaction-diffusion equations coupled with an ODE system. Several papers present results of existence and uniqueness of solution for the isolated bidomain model given by the equations (3.6) and the boundary conditions (3.8).

A first result for the bidomain model coupled with the FitzHugh-Nagumo ionic model, described above by the equation (3.12), is given in [CFS02]. This paper also deals with the microscopic model at the cell level and the proof relies on a reformulation of the system of equations in a non degenerated variational evolution inequality. For a simplified ionic model under the form

$I_{\text{ion}}(V_{\text{m}}, w) = I_{\text{ion}}(V_{\text{m}})$, the analysis is achieved in [BK06]. The paper [BCP09] shows the existence, uniqueness and regularity of a local solution for the bidomain model with a general ionic model by using the semi-group theory. It also shows the existence of a global in time solution for a large class of ionic models – including FitzHugh-Nagumo (3.12), Aliev-Panfilov (3.13) and Roger-McCulloch (3.14) – thanks to a compactness argument. At last, in [Ven09], the existence, uniqueness and the regularity of the solution are studied for the Luo-Rudy I ionic model.

All the mentioned papers only consider a model for the heart which is assumed to be decoupled from the external medium. In paper [BFGZ08] which is a common work with Miguel Fernández, Jean-Frédéric Gerbeau and Nejib Zemzemi, we have worked on the mathematical analysis of the coupled heart-torso system given by the equations (3.6), (3.9) and (3.10). The well-posedness of this problem is studied for a general class of ionic models with 2 variables including:

- FitzHugh-Nagumo model [Fit61, NAY62]:

$$I_{\text{ion}}(V_{\text{m}}, w) = kV_{\text{m}}(V_{\text{m}} - a)(V_{\text{m}} - 1) + w, \quad g(V_{\text{m}}, w) = -\epsilon(\gamma V_{\text{m}} - w); \tag{3.12}$$

- Aliev-Panfilov model [AP96]:

$$I_{\text{ion}}(V_{\text{m}}, w) = kV_{\text{m}}(V_{\text{m}} - a)(V_{\text{m}} - 1) + V_{\text{m}}w, \quad g(V_{\text{m}}, w) = \epsilon(\gamma V_{\text{m}}(V_{\text{m}} - 1 - a) + w); \tag{3.13}$$

- Roger-McCulloch model [RM94]:

$$I_{\text{ion}}(V_{\text{m}}, w) = kV_{\text{m}}(V_{\text{m}} - a)(V_{\text{m}} - 1) + V_{\text{m}}w, \quad g(V_{\text{m}}, w) = -\epsilon(\gamma V_{\text{m}} - w); \tag{3.14}$$

- a regularized version of Mitchell-Schaeffer model [MS03]. The usual model is given by:

$$\begin{aligned} I_{\text{ion}}(V_{\text{m}}, w) &= \frac{w}{\tau_{\text{in}}} V_{\text{m}}^2 (V_{\text{m}} - 1) - \frac{V_{\text{m}}}{\tau_{\text{out}}}, \\ g(V_{\text{m}}, w) &= \begin{cases} \dfrac{1-w}{\tau_{\text{open}}} & \text{if } V_{\text{m}} \leq v_{\text{gate}}, \\ \dfrac{-w}{\tau_{\text{close}}} & \text{if } V_{\text{m}} > v_{\text{gate}}. \end{cases} \end{aligned} \tag{3.15}$$

  The regularized version (which is given in [BFGZ08]) consists to modifying the definition of g in order to have a continuous fonction with respect to V_{m} at point v_{gate}.

Here, $0 < a < 1$, k, ϵ, γ, τ_{in}, τ_{out}, τ_{open}, τ_{close} and $0 < v_{\text{gate}} < 1$ are given positive constants. In these models, the potential is normalized with variations between 0 and 1.

The following theorem gives the global existence of solutions for the coupled heart-torso model. As in [BCP09] for the heart alone, we also get the uniqueness of solutions if the ionic current is given by FitzHugh-Nagumo model.

Theorem 3.1 *Let $\boldsymbol{\sigma}_{\text{i}}, \boldsymbol{\sigma}_{\text{e}} \in \boldsymbol{L}^\infty(\Omega_{\text{H}})$ be symmetric and uniformly definite positive, $w_0 \in L^2(\Omega_{\text{H}})$, $V_{\text{m}}^0 \in H^1(\Omega_{\text{H}})$ and $I_{\text{app}} \in L^2\big((0,T)\times\Omega_{\text{H}}\big)$ be given data. Assume that I_{ion} and g are given by* (3.12), (3.13), (3.14) *or a regularized version of* (3.15). *In $\Omega \stackrel{\text{def}}{=} \Omega_{\text{T}} \cup \overline{\Omega_{\text{H}}}$, we define u by:*

$$u \stackrel{\text{def}}{=} \begin{cases} u_{\text{e}} & \textit{in} \quad \Omega_{\text{H}}, \\ u_{\text{T}} & \textit{in} \quad \Omega_{\text{T}}. \end{cases}$$

Then, the system of equations (3.6), (3.9) *and* (3.10) *admits a weak solution $V_{\text{m}} \in L^\infty(0,T;H^1(\Omega_{\text{H}})) \cap H^1(0,T;L^2(\Omega_{\text{H}}))$, $w \in H^1(0,T;L^2(\Omega_{\text{H}}))$ and $u \in L^\infty(0,T;H^1(\Omega))$. Moreover, for the FitzHugh-Nagumo model* (3.12)*, the solution is unique.*

The proof of Theorem 3.1 is reported in [BFGZ08] and [Zem09, Part II]. It generalizes some of the arguments used for the analysis of the bidomain problem in [BK06, BCP09] to the case of the heart-torso coupling. The main idea consists of reformulating the bidomain system as a couple of degenerate reaction-diffusion equations and approximate the resulting heart-torso system by a suitably regularized problem in finite dimension. This problem is then analyzed through a Faedo-Galerkin/compactness procedure and a specific treatment of the non-linear terms. The heart-torso coupling is handled through an adequate definition of the Galerkin basis. Compared to the models (3.12) and (3.14), the regularized version of Mitchell-Schaeffer model (3.15) has a different structure and the proof slightly differs.

The original Mitchell-Schaeffer model (without any regularization) (3.15) has been considered in the recent paper [KM]. Since the function g is discontinuous, the proof uses Filippov theory which replaces the ODE by a differential inclusion involving a multi-valued extension of the non-linearity. The paper [KM] considers the heart alone with the monodomain model but it should be possible to generalize the proof to the bidomain model coupled with the torso model.

3 Numerical simulations of ECGs - [BCF+10]

Paper [BCF+10] is a work in common with Miguel Fernández, Jean-Frédéric Gerbeau and Nejib Zemzemi which has been achieved during the PhD of Nejib. The medical doctor Serge Cazeau gave us precious comments on the numerical results to improve them. This work is also presented in the book chapter [BFGZ11]. A preliminary version of these results was presented in the proceeding [BFGZ07].

As explained in Section 2.2, since ECGs are the main tool to observe the electrical functioning of the heart, it is capital to be able to reproduce them through numerical simulations. First of all, it allows to validate our numerical simulations and to submit our results to the critical evaluation of a medical doctor. Moreover, computer based simulations of the ECG can be a valuable tool to compare different models and to improve the knowledge of ECG. At last, it is a necessary step before addressing inverse problems based on ECG measurements.

The main objective of [BCF+10] was to introduce a model able to produce realistic healthy ECGs and some pathological ECGs. While the numerical simulation of ECGs has been addressed in many works, only a very few [TDP+04, PDV09, BCF+10, CGS] provide meaningful simulations of the 12-lead ECG. In [TDP+04, PDV09], simulations rely on either a monodomain approximation or a heart-torso decoupling approximation and a multi-dipole cardiac source representation. Our paper [BCF+10] was the first to show realistic 12-lead ECG for a model based on partial differential equations (PDE) and a full heart-torso coupling.

In a classical way, the model is based on the system of equations (3.6) and (3.9). To complete this model, several aspects have to be elucidated: ionic model, heart–torso transmission conditions, cell heterogeneity, His bundle modeling, anisotropy... and it is capital to understand the influence of all these important features on the cardiac electrical front.

Our study is mainly organized in two steps: in a first stage, we present reference simulations, based on specific choices for these modeling issues. These simulations allow to get realistic ECGs in the healthy case and in some pathological cases. In a second stage, alternative modeling choices are made in order to assay methodically the influence of the modeling assumptions on the ECG. The objective of this step is to see to what extent the choices of the reference simulations are relevant.

Let us first present the modeling hypotheses made to get our reference simulations (see [BCF+10] for the details):

- the cardiac cell membrane dynamics are based on the Mitchell-Schaeffer ionic model (3.15).
- cells are heterogeneous in terms of *Action Potential Duration* (APD), which transmurally varies within the left ventricle. In practice, this amounts to consider a varying parameter τ_{close} in (3.15) which takes three different values in the left ventricle and another value in the right ventricle.
- the heart conductivities are anisotropic;
- the fast conduction system (His bundle and Purkinje fibers) is modeled by initializing the activation with a time-dependent external volume current I_{app}, acting on a thin subendocardial layer of left and right ventricles.
- the coupling between the heart and the torso is strong (3.10);
- the torso geometry contains three regions modeled by different values of $\boldsymbol{\sigma}_{\mathrm{T}}$: the lungs, the bones and the remaining extracardiac tissues (see Figure 3.3).

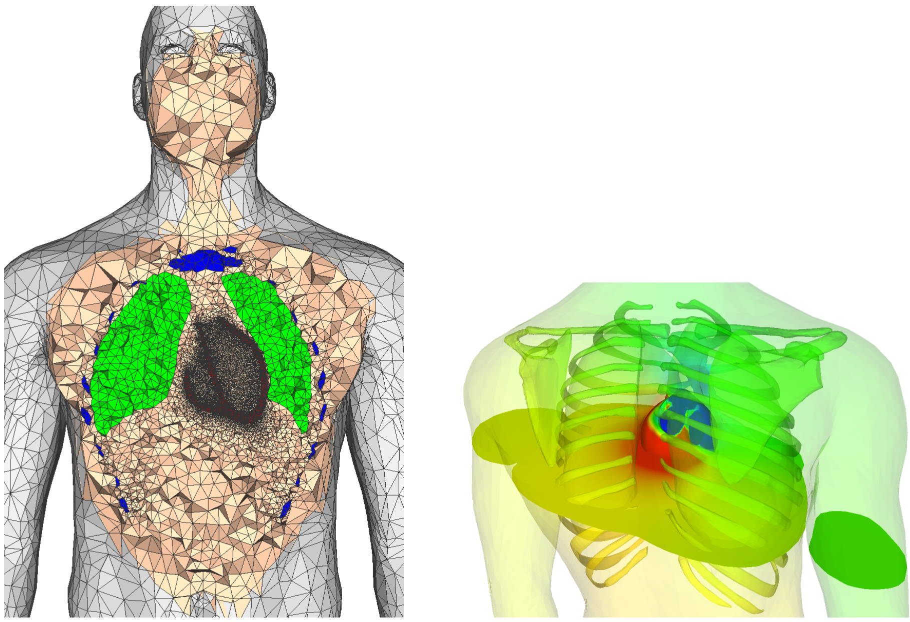


Figure 3.3: *Left*: cut view of the heart-torso computational mesh. Heart (red), lungs (green), bone (blue) and remaining tissue (apricot). *Right*: posterior view and cut plane of the torso and heart potentials at time $t = 10$ ms.

The system is discretized in time by combining a second order implicit scheme (backward differentiation formulae) with an explicit treatment of the ionic current and the resulting system is discretized in space using finite elements. The heart-torso strong coupling is solved, using a partitioned iterative procedure based on the Dirichlet-Neumann algorithm combined with an acceleration strategy (based on a relaxation parameter computed thanks to an Aitken formula). For references on the numerical methods, parallel approaches are described in details in [CFP04], [SPM08] and

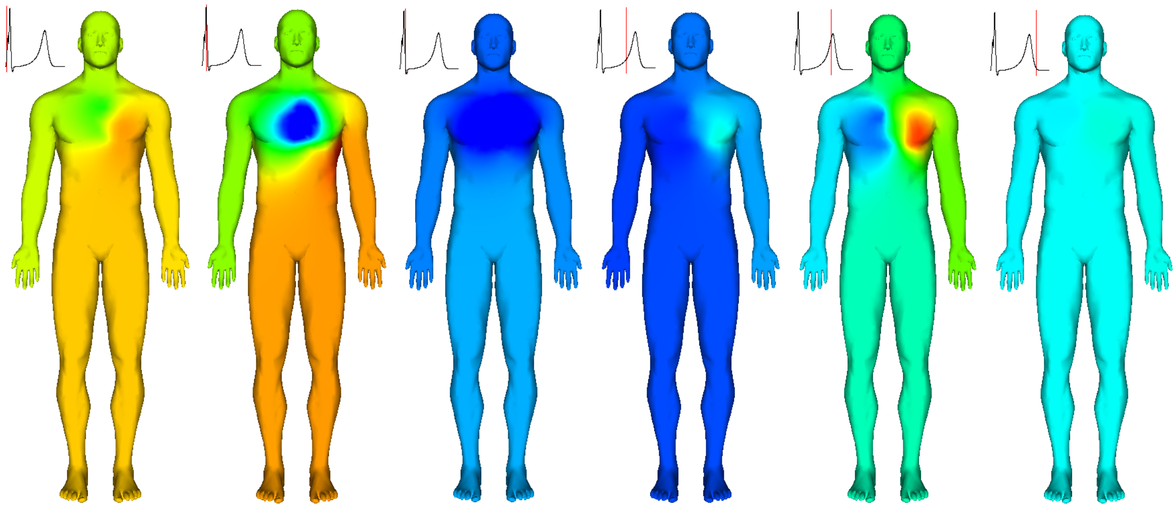


Figure 3.4: Snapshots of the body surface potentials at times t = 10, 32, 40 ms (depolarization) and t = 200, 250 and 310 ms (repolarization), from left to right.

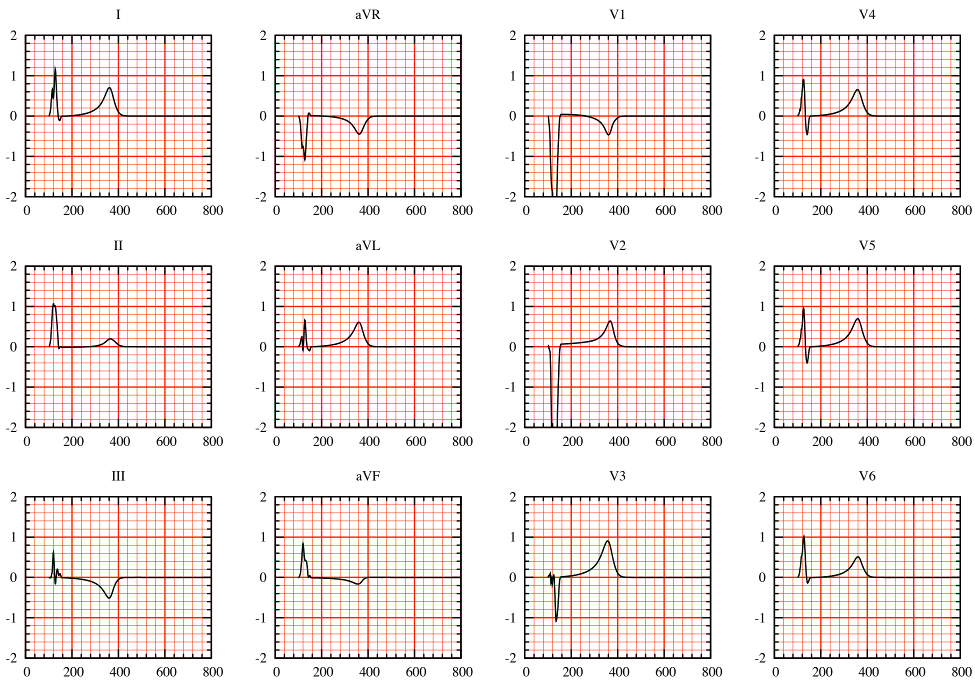


Figure 3.5: Simulated normal 12-lead ECG signals.

[GGPV11], preconditioning techniques in [PS08] and [GGMN+09]. For more references, we refer to the reviews [LBG+03] and [VWdSP+08].

Figure 3.4 shows some snapshots of the simulated body surface potentials. The corresponding 12-lead ECG signals are given in Figure 3.5. Despite some minor flaws, the comparison with Figure 3.1 (right) shows that the obtained numerical ECGs have the correct amplitudes, shapes and polarities, in all the twelve standard leads. Numerical simulations have also been carried out for some pathological conditions like left or right bundle branch blocks (see [BCF+10] or

[Zem09] for details). It is worth noting that the numerical ECG signals satisfy the typical criteria used by medical doctors to detect the pathology, and this without any recalibration of the model parameters besides the natural modifications needed to model the disease (*i.e.* delayed activation in the right or the left ventricle). These simulations constitute a breakthrough in the numerical simulations of ECGs with partial differential equations.

In a second part, the ECGs obtained with alternative modeling choices are compared to the reference ECG given by Figure 3.5. The numerical results show that cell heterogeneity and fiber anisotropy have an important impact on the ECG and, therefore, cannot be neglected (for a precise study on the effect of anisotropy, we refer to [CFPT05]). Taking into account the cell heterogeneity in order to translate the variations of the APD is an important feature of the reference model: it allows to obtain the T-wave with a correct polarity. On the other hand, the monodomain approximation seems sufficient to recover realistic ECGs.

We also noticed that, if the strong coupling conditions (3.10) are replaced by the weak coupling conditions (3.11), the amplitude of the ECG differs significantly (it is approximately multiplied by 2) but the shape of the signal, especially in the healthy case, is weakly perturbed. Thus, it may be a reasonable choice to get qualitatively correct ECGs in a reduced time (this simplification allows to divide the computational time by about 10).

The coupling conditions (3.10) correspond to a perfect electrical coupling between the heart and the surrounding tissues. We also considered more general coupling conditions by assuming that the pericardium (the double-walled sac which contains the heart) can induce a resistor-capacitor effect. The strong coupling conditions become:

$$\begin{cases} R_p \boldsymbol{\sigma}_{\mathrm{T}} \nabla u_{\mathrm{T}} \cdot \mathrm{n} = R_p C_p \dfrac{\partial (u_{\mathrm{e}} - u_{\mathrm{T}})}{\partial t} + (u_{\mathrm{e}} - u_{\mathrm{T}}) & \text{on} \quad (0,T) \times \Sigma \\ \boldsymbol{\sigma}_{\mathrm{e}} \nabla u_{\mathrm{e}} \cdot \mathrm{n} = \boldsymbol{\sigma}_{\mathrm{T}} \nabla u_{\mathrm{T}} \cdot \mathrm{n} & \text{on} \quad (0,T) \times \Sigma. \end{cases}$$

In this expression, R_p and C_p are respectively the resistance and the capacitance of the pericardium. Notice that we recover the perfect coupling by setting $R_p = 0$. These conditions may correspond to a pathology affecting the pericardial sac (e.g. the pericarditis) and we observed that the the capacitor induces a relaxation effect and distorts the signal while the resistance reduces the amplitude.

Let us mention that the heart geometry only includes the ventricles. This prevents from computing the P-wave of the ECG. In the recent paper [CGS], Annabelle Collin, Jean-Frédéric Gerbeau and Elisa Schenone have reproduced full cycle ECGs for a geometry of the heart including atria. Moreover, ECGs for additional pathologies have been displayed and gave convincing results, which confirm the predictive features of the ECG simulator.

The ionic Mitchell-Schaeffer model which has been used in these simulations has the advantage to reproduce a realistic action potential. Moreover, each parameter of the model has a physiological interpretation, so that their value can be tuned in a consistent way. In [CGS], the more complex phenomenological MV model introduced in [BOCF08] is used. Computing numerical ECGs with MV model leads to slightly improved results.

To end with, I mention that I am currently working with J.-F. Gerbeau and F. Raphel on the modeling and numerical simulation of the electrical activity of a cardiac cell layer cultivated in a device called Micro-Electrode Array (MEA). This device is an array of electrodes which gives spatiotemporal measurements of the cardiac potential. It is mostly used in pharmacology for drug testing and is an alternative to animal-based experimentations. This work is achieved in the framework of the LabCom CardioXcomp leaded by Jean-Frédéric Gerbeau and funded by the ANR with a partnership with the company Notocord, an editor of medical software for data acquisition and analysis.

Reproducing the measurements made by MEA raises many issues: modeling the device, the cardiac cells (induced pluripotent stem cells) and the drugs, reproducing the variability of the measurements. In this context, our long-term objective is twofold: first, pharmacologists could take advantage of the prediction property of our simulations to plan experiments. Next, integrating inverse problem techniques could allow to interpret the measurements made by MEA: for a pharmacologist, knowing the specific ionic channels a drug will act on is an information much more valued than observing the direct effect of the drug on the measurements. We have started to address this question with sequential methods like Unscented Kalman Filter.

4 Inverse problems

In the pathological cases considered in [BCF+10], we apply modifications which model some diseases and then describe the effect on the ECG. This situation where we try to reproduce the effects of a known cause is referred to as the forward problem. To go further and to exploit the potential of numerical studies in terms of medical or clinical applications, it is essential to address the inverse problems: knowing the effect, is it possible to determine the cause ? This situation corresponds to the one of the medical doctor who has access to measurements and try to trace back to an information on the cardiac electrical front to make a diagnosis.

In this part, I will describe studies related to the inverse problems in electrocardiology. First, a theoretical study presented in Subsection 4.1 gives some stability inequalities for the identification of parameters in the bistable equation. Subsection 4.2 is a brief introduction to the classical inverse problem in electrocardiology. At last, in Subsection 4.3, we are interested by the numerical identification of some parameters of the model from measurements given by ECG.

4.1 Theoretical study - [BGO09]

This study is a common work with Céline Grandmont and Axel Osses. We are interested in the identification of the nonlinear term in the bistable equation from internal measurements. More precisely, we introduce the bistable equation:

$$\begin{cases} \partial_t V_\mathrm{m} - \Delta V_\mathrm{m} & = f(V_\mathrm{m}, x) & \text{in } Q \\ \nabla V_\mathrm{m} \cdot n & = 0 & \text{on } (0,T)\times\partial\Omega, \\ V_\mathrm{m}(0) & = V_{\mathrm{m},0} & \text{in } \Omega, \end{cases} \tag{3.16}$$

where $Q = (0,T)\times\Omega$ and

$$f(V_\mathrm{m}, x) = a(x)V_\mathrm{m}(1 - V_\mathrm{m})(V_\mathrm{m} - \alpha(x)).$$

The bistable equation naturally appears if we consider the monodomain model for an insulated heart and if we simplify the expression of the ionic current by omitting the ionic variable in the phenomenological models (3.12), (3.13) or (3.14).

In this equation, we want to identify the parameters a and α and assume that we have measurements of V_m in $(0,T)\times\omega$ where ω is an arbitrary nonempty open subset of Ω. These parameters are assumed to satisfy:

$$a \in W^{1,\infty}(\Omega), \quad \exists\, a_0, a_1 \in \mathbb{R} \text{ such that } 0 < a_0 \leqslant a(x), \forall\, x \in \Omega \text{ and } \|a\|_{W^{1,\infty}(\Omega)} \leqslant a_1, \tag{3.17}$$

$$\alpha \in L^\infty(\Omega), \quad \exists\, \alpha_0, \alpha_1 \in \mathbb{R} \text{ such that } 0 < \alpha_0 \le \alpha(x) \le \alpha_1 < 1, \forall x \in \Omega. \tag{3.18}$$

A list of other related inverse problems of this kind is presented in [EEK05]. To obtain stability estimates, we first proved a new global Carleman inequality. Like for the heat equation, this inequality involves a function ψ in Ω such that

$$\psi \in C^2(\overline{\Omega}),\ \psi > 0 \text{ in } \Omega,\ \psi = 0 \text{ on } \partial\Omega,\ |\nabla\psi| > 0 \text{ in } \overline{\Omega \setminus \omega'},$$

where $\omega' \subset\subset \omega$ is a nonempty open set. We then define, for all $\lambda > 0$ and $s > 0$ the weights:

$$\varphi(\psi) = \frac{e^{2\lambda\|\psi\|_\infty} - e^{\lambda\psi(x)}}{t(T-t)}, \quad \eta(\psi) = \frac{e^{\lambda\psi(x)}}{t(T-t)}, \quad \rho(\psi) = e^{-s\varphi(\psi)}, \tag{3.19}$$

the weighted integral:

$$I_\psi(V_{\mathrm{m}}) = \iint_Q \rho^2 \Big(\frac{1}{s\eta}(|\partial_t V_{\mathrm{m}}|^2 + |\Delta V_{\mathrm{m}}|^2) + |V_{\mathrm{m}}|^6 + s\lambda^2\eta|\nabla V_{\mathrm{m}}|^2 + s^3\lambda^4\eta^3|V_{\mathrm{m}}|^2 + s^2\lambda^2\eta^2|V_{\mathrm{m}}|^4 \Big)\, dx\, dt$$

and the weighted local function of observations:

$$N_{T,\omega,\psi}(V_{\mathrm{m}}) = \iint_{(0,T)\times\omega} \rho^2 (s^3\lambda^4\eta^3|V_{\mathrm{m}}|^2 + s^2\lambda^2\eta^2|V_{\mathrm{m}}|^4)\, dx\, dt.$$

The Carleman inequality is then the following:

Lemma 3.2 *There exists $\overline{\lambda}$ and C only depending on Ω and ω such that, for any $\lambda \geqslant \overline{\lambda}$ and $s \geqslant \overline{s} = C(\Omega, \omega, T, a_0, a_1)e^{2\lambda\|\psi\|_\infty}$, for any $V_{\mathrm{m},0} \in L^2(\Omega)$, the solution V_{m} of (3.16) satisfies*

$$I_\psi(V_{\mathrm{m}}) \leqslant C N_{T,\omega,\psi}(V_{\mathrm{m}}). \tag{3.20}$$

This inequality is shown following the same scheme as for the heat equation [FI96] except that the nonlinear terms have to be treated carefully. Since we consider Neumann boundary conditions, we also have to deal with the boundary integrals. The idea (we refer for instance to [FI96] and [FCGBGP06]) is to do the calculations for ψ and also for $-\psi$ and then sum up the two inequalities in order to cancel the boundary integrals.

This Carleman inequality proved directly on the nonlinear problem is a result in itself and allows for instance to show the unique continuation property for the equation (3.16):

Corollary 3.3 *Let V_{m} be a weak solution of (3.16). If $V_{\mathrm{m}} = 0$ in $(0,T) \times \omega$, then $V_{\mathrm{m}} = 0$ in Q.*

Then, using the classical Bukhgeim-Klibanov method [BK81], we show Lipschitz stability inequalities on the parameters a and α:

Theorem 3.4 *Let $V_{\mathrm{m},0} \in H^2(\Omega)$ and $\overline{V}_{\mathrm{m},0} \in H^4(\Omega)$, a and $\overline{a}$ satisfy (3.17) and α and $\overline{\alpha}$ satisfy (3.18). We denote by $(p, \overline{p}) = (a, \overline{a})$ or $(\alpha, \overline{\alpha})$ and by $\overline{V}_{\mathrm{m}} = V_{\mathrm{m}}(\overline{V}_{\mathrm{m},0}, \overline{p})$ and $V_{\mathrm{m}} = V_{\mathrm{m}}(V_{\mathrm{m},0}, p)$ the corresponding solutions of* (3.16)*. Let us consider t_0 such that $0 < t_0 \leqslant T/2$ and let us assume that*

$$\left| \frac{\partial f}{\partial p}(\overline{v}(x, t_0)) \right| \ge r_0 > 0 \text{ for all } x \in \Omega. \tag{3.21}$$

Then there exists $C > 0$ such that

$$\|p - \overline{p}\|_{L^2(\Omega)} \le C N_{T,\omega}(V_{\mathrm{m}} - \overline{V}_{\mathrm{m}}),$$

where $N_{T,\omega}(V_{\mathrm{m}}) = \|V_{\mathrm{m}}\|_{H^1(0,T;L^2(\omega))} + \|V_{\mathrm{m}}\|^2_{L^4(0,T;L^4(\omega))} + \|V_{\mathrm{m}}(t_0)\|_{H^2(\Omega)} + \|V_{\mathrm{m}}(t_0)\|^3_{L^6(\Omega)}$.

Let us notice that, as for the heat equation, we need local measurements in the whole time interval $(0,T)$ and we also need global measurements in Ω at an arbitrary time t_0. The assumptions of this result may be relaxed in the sense that, if inequality (3.21) does not hold on the whole domain Ω, we will deduce an estimate of $p-\overline{p}$ on the subset where $\frac{\partial f}{\partial p}(\overline{v}(\cdot,t_0))$ stays away from 0.

To show this inequality, we can use the Carleman inequality (3.20). Nevertheless, due to the regularity of the solution (with the hypotheses of Theorem 3.4, V_{m} belongs to $L^\infty(Q)$ and $\overline{V}_{\mathrm{m}}$ belongs to $W^{1,\infty}(0,T;L^\infty(\Omega))$), it is sufficient to use the classical Carleman inequality for the heat equation since the nonlinear terms V_{m}^2 and V_{m}^3 of (3.16) can be handled like the linear term V_{m}.

To finish this subsection, I would like to mention an ongoing work with Elisa Schenone. We are working on the extension of these results to the monodomain equation coupled to an ODE corresponding to the evolution of the ionic variable:

$$\begin{cases} \partial_t V_{\mathrm{m}} - \Delta V_{\mathrm{m}} & = f(V_{\mathrm{m}}, w) & \text{in } Q, \\ \partial_t w & = g(V_{\mathrm{m}}, w) & \text{in } Q, \\ \nabla V_{\mathrm{m}} \cdot n & = 0 & \text{on } (0,T)\times\partial\Omega, \\ V_{\mathrm{m}}(0) & = V_{\mathrm{m},0} & \text{in } \Omega, \\ w(0) & = w_0 & \text{in } \Omega. \end{cases}$$

If we consider the ionic models (3.12), (3.13) or (3.14), we see that the solution w of the second equation can be expressed explicitly with respect to V_{m}. Then, if we put this expression in the first equation, we get a reaction-diffusion equation with a memory term.

As noticed in [CSRZ14] which considers a similar problem, it is not possible to derive a global observability inequality for this coupled system. This is due to the presence of the ODE which induces a lack of propagation in space-like direction. In paper [CSRZ14] which studies the controllability of a problem which couples a linear parabolic equation and an ODE, the authors overcome this lack of controllability by considering a variable control with a support which covers the whole domain. Despite this problem, the explicit writing of w with respect to V_{m} will allow to get directly stability estimates for the identification of some parameters of the model.

Another interesting question (closely linked to the problem presented in the next subsection) would be to couple the model of the heart with the model of the torso and to see whether measurements on the external boundary (corresponding to ECG) allow to identify parameters involved in the heart model. Having in mind the results presented in Chapter 2, it should not be possible to have better than logarithmic estimates, except if we assume that the parameters are piecewise constant.

4.2 The inverse problem in electrocardiography

What is commonly called *the inverse problem in electrocardiography* is the reconstruction of the cardiac potential on the epicardium from measurements at the body surface. In mathematical terms, using again the notation of Section 2, this can be formulated as follows: in the torso, if we denote by $g = u_{\mathrm{T}}|_\Sigma$ the function that we want to recover and by $u_{\mathrm{T}}(g)$ the solution of

$$\begin{cases} -\operatorname{div}(\boldsymbol{\sigma}_{\mathrm{T}}\nabla u_{\mathrm{T}}) = 0 & \text{in} \quad (0,T)\times\Omega_{\mathrm{T}} \\ \boldsymbol{\sigma}_{\mathrm{T}}\nabla u_{\mathrm{T}}\cdot \mathrm{n} = 0 & \text{on} \quad (0,T)\times\Gamma_{\mathrm{ext}} \\ u_{\mathrm{T}} = g & \text{on} \quad (0,T)\times\Sigma \end{cases} \tag{3.22}$$

the inverse problem is to find a function $g \in L^2(0,T;H^{1/2}(\Sigma))$ such that

$$u_{\mathrm{T}}(g) = u_{\mathrm{m}} \quad \text{on} \quad (0,T) \times \Gamma$$

where $\Gamma \subset \Gamma_{\mathrm{ext}}$ is the part of the body surface where measurements are available and $u_{\mathrm{m}} \in L^2(0,T;H^{1/2}(\Gamma))$ corresponds to the recorded potential on $(0,T) \times \Gamma$. This problem is referred as the Cauchy problem. It appears in many application domains and has also been presented in Chapter 2.

Since Hadamard, this problem is known to be ill-posed in the sense that small perturbations on the measurements may induce large fluctuations of the inverse solution: this is due to the fact that the stability inequality is of logarithmic type. We refer to [ARRV09] for a general presentation and to [BMN10] which completes these results by studying the stability of computing the transmembrane potential from ECG recordings.

As proved in the papers [Sin07] and [Bou13] which deal with similar problems, if the function to identify g lives in a space of finite dimension, the inverse problem behaves better with a Hölder rate but the constant depends on the dimension of the space with an exponential increase. Thus, the numerical discretization introduces a regularization which is however not sufficient to lead to a well-posed problem.

A classical way to circumvent these problems is to regularize the problem thanks to a Tikhonov term. Then, the objective is to find a function g which minimizes the cost function

$$\|u_{\mathrm{T}}(g) - u_{\mathrm{m}}\|^2_{L^2((0,T)\times\Gamma)} + \lambda \|g\|^2$$

Different choices of norms $\|\cdot\|$ have been considered in the literature (L^2-norms, second-order regularization in space, first order regularization in time). We refer to [SLC$^+$06], [CFPCS06] and [Sch14] for a detailed presentation of this method and of other regularization methods like TSVD. Of course, the choice of the norm and the value of the parameter λ has a strong impact on the solution of this problem and knowing how to choose them is a difficult question.

Another way to regularize the problem is to reduce the complexity of the function to identify by simplifying the representation of the source: for instance in [SLC$^+$06], the authors deal with the identification of a set of vectors which represents the cardiac source or in [CFPCS06], the authors want to identify the position of the cardiac wavefront.

If we want to identify a specific cardiac pathology from ECG, it is sufficient to identify parameters which corresponds to this pathology. By this way, the ill-posed Cauchy problem is replaced by the identification of a finite number of parameters involved in the cardiac model and we may hope that this problem is better posed. To be able to proceed this way, it is necessary to have a direct model which represents accurately the evolution of the source i.e. the cardiac potential in the heart. In other words, the function g in (3.22) is not arbitrary and is the restriction to Σ of a cardiac potential described by a cardiac model where some parameters are unknown. This is the point of view which has been chosen in the paper presented in Subsection 4.3. Even if this approach seems natural, it is not classical especially because it requires a direct model able to reproduce realistic ECG. In our work, we have used the direct model presented in [BCF$^+$10].

Let us notice that, compared to the Cauchy problem, this problem is far heavier to solve because the model of the source is complex and we have to face with a nonlinear problem.

4.3 Numerical identification of parameters - [BGS12]

In this part, I present the paper [BGS12] which has been achieved with Jean-Frédéric Gerbeau and Elisa Schenone during the PhD of Elisa that I co-supervised with Jean-Frédéric. Our main objective is to identify thanks to numerical simulations some parameters of the cardiac model from measurements given by ECG.

The model is given by the bidomain model (3.6) completed by the Mitchell-Schaeffer ionic model (3.15) in the heart, the Laplace equation (3.9) in the torso and these equations are weakly coupled through the boundary conditions (3.8). Thus, if we denote by $\theta \in \mathbb{R}^n$ the vector of parameters that we want to identify and if we denote by $(u_\mathrm{e}(\theta), u_\mathrm{T}(\theta))$ the solution of system (3.6)-(3.15)-(3.9)-(3.8), we want to minimize the cost function

$$J(\theta) = \|u_\mathrm{T}(\theta) - u_\mathrm{m}\|^2_{L^2((0,T)\times\Gamma)}$$

with respect to θ which belongs to an admissible subset $\mathcal{I}$ of $\mathbb{R}^n$. We considered different kinds of parameters with a special interest on the identification of an infarcted zone. In our simple ionic model (3.15), this pathology is modeled by dividing by 10 the parameter τ_out in the infarcted region. The parameter to identify in this case is the center of the infarcted zone which is assumed to be a ball of known radius.

A classical way to solve such minimization problem is to use a descent method based on the computation of the gradient. To do so, it is necessary to compute the adjoint of the tangent linear model. This choice of resolution would lead to very heavy computational efforts. To avoid this, we have chosen an inverse procedure based on a genetic algorithm. This approach offers many advantages: it can easily be run in parallel and it does not need the gradient of the cost function. It is a global optimization method which consists of following the evolution of a population of elements corresponding to a set of values of the parameters. The population is regenerated several times. At each generation, the cost function is evaluated for each element of the population and the population evolves from a generation to another following three stochastic principles: selection (promote the elements of the population whose value by the cost function is small), crossover (create from two elements of the population two new elements by doing a barycentric combination of them with random and independent coefficients), mutation (replace an element of the population by a new one randomly chosen in its neighborhood). To speed up this algorithm, many evaluations of the cost function are performed using a surrogate model. This model consists of approximating the value of J by a Radial Basis Functions interpolation based on previously computed exact evaluations. The total number N_{ex} of exact evaluations is fixed and the number of exact evaluations decreases at each generation. The interested reader is referred for example to [Gol89] for more details about this algorithm.

The main flaw of genetic algorithms is to require a large number of evaluations of the direct problem, even if many evaluations are avoided with the surrogate model strategy. To keep the computational time reasonable, we used a reduced order model based on Proper Orthogonal Decomposition (POD) in the optimization loop.

POD is a method used to derive reduced models by projecting the system onto subspaces spanned by a basis of elements that contains the main features of the expected solution. We briefly recall the method here and refer to [KV01], [RP04] for more details. To generate the POD basis associated with a precomputed solution $u = (V_\mathrm{m}, u_\mathrm{e})$ of the discretized problem of dimension n, we make a first numerical simulation (or a set of simulations) and keep some snapshots $u(t_k)$, $1 \leqslant k \leqslant p$ which correspond to the solution at some specific times. Then a singular value decomposition (SVD) of the matrix $B = (u(t_1), \dots, u(t_p))$ is performed $B = USV'$, where $U \in \mathbb{R}^{N,N}$ and $V \in \mathbb{R}^{p,p}$ are orthogonal matrices, $S \in \mathbb{R}^{N,p}$ is the matrix of the singular values ordered by decreasing order, and $N \geqslant p$ is the dimension of the Galerkin basis of the finite element method. The N_modes first

POD basis functions $\{\Psi_i\}_{1\leqslant i\leqslant N_{\text{modes}}}$ are then given by the N_{modes} first columns of U and the POD Galerkin problem is solved by looking for a solution of the type

$$u = \sum_{i=1}^{N_{\text{modes}}} \alpha_i(t)\Psi_i.$$

The $N \times N$ sparse system of the finite element method is thus replaced by a full system of size $N_{\text{modes}} \times N_{\text{modes}}$ with the POD method. To give a rough idea, it is generally possible to get a good accuracy for the problems at hand with $N_{\text{modes}} \approx 100$. With the time scheme used in this work, the matrix is constant over the time, since all the nonlinearities are treated explicitly. The matrix is therefore projected on the POD basis and factorized only once at the beginning of the computation. As a consequence, in our simulations, the reduced order model resolution is about one order of magnitude faster than the full order one.

To use the reduced model in parameter identification problems, a critical difficulty has to be faced: a POD basis generated from a solution obtained with a given set of parameters may be inaccurate to approximate a solution obtained from another set of parameters. The issue of the stability of reduced basis with respect to parameters perturbation is the topic of active researches (we refer to [GMNP07], [AF08] among many other studies). In [BGS12], we first studied on the direct problem whether the full model can be accurately approximated by the reduced model based on POD and we proposed different approaches to compute the POD basis when the parameters vary. These approaches are simple but seem quite efficient for the considered problems.

We observed that it is in general possible to use a POD basis generated with an arbitrary given set of parameters to approximate a solution for another set of parameters provided that the variation of the parameter does not induce a large spatial perturbation of the cardiac wavefront. In particular, if the parameters which vary are parameters of the model (for instance ionic parameters in the equation (3.15) or χ_{m} in (3.6)), this basic approach gives good results as long as the parameters values stay in an acceptable set. On the contrary, this approach is totally inadequate to approximate the full model when the parameter which varies corresponds to the position of the initial stimulation [BG11] or to the position of an infarcted area (we refer to Figures 7 and 8 in [BGS12]). In these cases, the parameter variation has too much impact on the spatial evolution of the wavefront. To circumvent this problem, a natural strategy consisting of precomputing several POD bases with different sets of parameters, or a single POD basis from different experiments, proved to be satisfactory in the cases we have considered.

The reduced-order model with the adequate strategy to generate the POD basis is then used to solve parameter identification problems with the genetic algorithm. Let us detail the results obtained in the case of an infarcted region. They can be summarized by Figures 3.6 and 3.7.

First, a simulation of the complete direct model with an infarcted area is run (Figure 3.7 Left) and allows to get a reference ECG (green line in Figure 3.6). The location of the pathology represents a transmural myocardial anterior infarction. Then, the optimization procedure based on genetic algorithms is run to recover the location of the center of this infarction area. The cost function corresponds to the discrepancy between the ECG obtained for a given value of the center of the infarction area and the reference ECG. The position of the infarction given by the algorithm is reported in Figure 3.7 Right and the obtained ECG corresponds to the blue line in Figure 3.6. The identified infarcted region is actually very close to the reference ECG. Let us mention that all the simulations presented in this work are based on synthetic data. In [BG11], perturbations on the reference ECG by a noise are applied and the algorithm still gives good results.

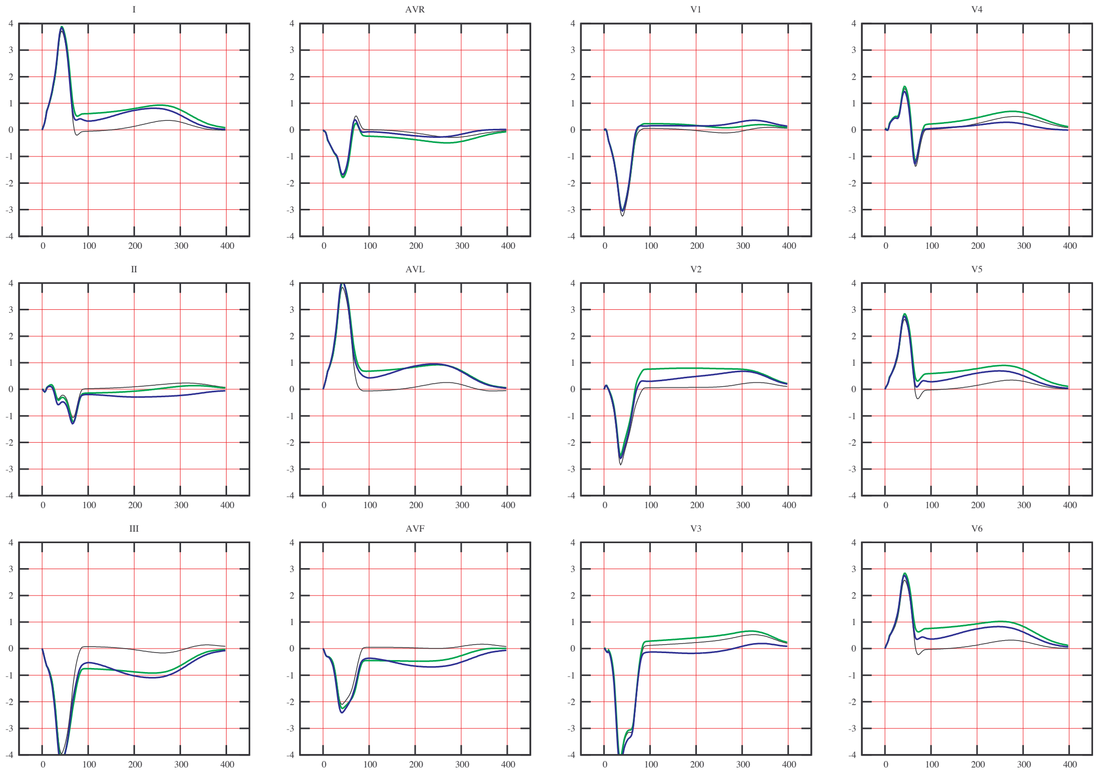
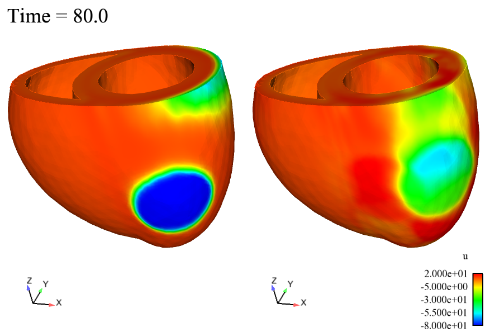
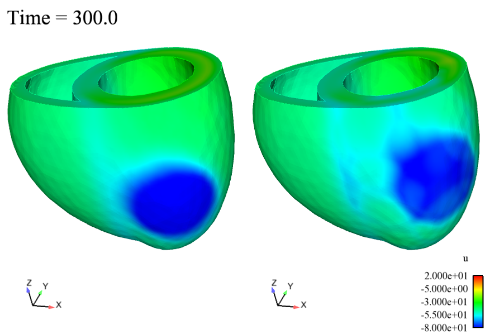


Figure 3.6: Simulated ECG with an infarction area: green line represents the simulated reference ECG and blue line the ECG corresponding to the infarcted center found with the resolution of a genetic algorithm. Black line gives the healthy reference case.



(a) t = 80ms



(b) t = 300ms

Figure 3.7: Left: transmembrane potential calculated in the reference case solved with the full model. Right: transmembrane potential of the solution found with the genetic algorithm obtained with the POD.

For an alternative approach to tackle this problem, we refer to [NLT07] which uses level set techniques. An interesting possibility to investigate could be to enrich the cost function giving more weight to the ST deviation, as suggested in [GPF+04].

As a conclusion, a natural strategy consisting of precomputing several POD bases with different sets of parameters, or a single POD basis from different experiments, proved to be satisfactory in the cases we have considered. Nevertheless, this solution requires an important off-line effort which makes it difficult to apply with more than a few parameters. A new reduced order algorithm based on the Approximated Lax Pairs [GL14] is proposed in [GLS]. The main advantage of this method is that the reduced basis on which the solution is searched is not based on pre-computed simulations and evolves in time according to the problem dynamics. This approach is thus particularly interesting for the identification of parameters.

5 Impact of noise on the electrical activity of a cardiac tissue - [BGT]

5.1 Introduction

In this study which has been achieved with Alexandre Génadot and Michèle Thieullen, we consider the isolated monodomain problem driven by a stochastic term referred to as noise:

$$\left\{\begin{array}{lll} \mathrm{d}V_{\mathrm{m}} & = \big(\nu\Delta V_{\mathrm{m}} + \frac{1}{\epsilon} f(V_{\mathrm{m}}, w)\big)\mathrm{d}t + \sigma \mathrm{d}W & \text{in } (0,T)\times D \\ \mathrm{d}w & = g(V_{\mathrm{m}}, w)\mathrm{d}t & \text{in } (0,T)\times D \end{array}\right. \tag{3.23}$$

where D is a regular domain of $\mathbb{R}^2$. In this system, f and g are fixed by the chosen ionic model, ϵ corresponds to the timescale between the fast variable V_{m} and the slow variable w, W is a Wiener process whose properties will be specified later and σ corresponds to the noise intensity. In this study, we consider that the noise acts on the first equation. By this way, we assume that fluctuations directly affect the transmembrane potential contrary to conductance-based noises which correspond to a noise acting on the second equation.

Considering noise for neuron models is very common since it is well accepted that noise can affect the nervous-system function. These studies are usually made on stochastic differential equations which model the evolution of the potential in the nervous cell. More rarely, studies deal with stochastic partial differential equations. For instance, there are studies on the evolution of the potential through an axon modeled by the one-dimension cable partial differential equation completed by FitzHugh-Nagumo model [Tuc08] or Hodgkin-Huxley model [TJ10]. In these works, the noise is white in time and in space: it satisfies $\forall\, h,\, k \in \mathcal{D}(\mathbb{R}_+ \times \mathbb{R})$

$$\mathbb{E}(< \dot{W}, h >< \dot{W}, k >) = \int_0^{+\infty}\int_{\mathbb{R}} h(t,x)k(t,x)\,dx\,dt$$

which means that the noise is uncorrelated in time and in space.

In dimension greater that 1, PDEs such as Allen-Cahn equation (corresponding to the first equation of (3.23) with $f(V_{\mathrm{m}}, w) = V_{\mathrm{m}} - V_{\mathrm{m}}^3$) or more simply heat equation driven by white noise in time and space are known to be ill-posed [Wal86]. More precisely, in [HRW12], it is proved that the solution of Allen-Cahn equation with a mollified space-time white noise weakly converges to the trivial solution 0 when the mollifier is removed. This confirms numerical results [RNT12] which show that the discretized solution of the problem does not tend to the physically meaningful limit.

In our study, the noise is assumed to be white in time and colored in space. More precisely, let Q be a given linear bounded non-negative symmetric operator. We assume that Q is an operator with kernel: there exists a symmetric and non-negative function q defined on $D \times D$ such that

$$\forall \phi \in L^2(D), \forall x \in D, \quad Q\phi(x) = \int_D \phi(y) q(x,y) \mathrm{d}y.$$

We also assume that Q is a class trace operator. Then we can define a Q-Wiener process $(W_t^Q,\ t \in \mathbb{R}^+)$ on $L^2(D)$ (we refer to [PZ07] for the precise statement of this result). This process satisfies in particular: $\forall\, h,\ k \in \mathcal{D}(\mathbb{R}_+ \times \mathbb{R}^2)$

$$\mathbb{E}(< \dot{W}^Q, h >< \dot{W}^Q, k >) = \int_0^{+\infty} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} h(t,x) k(t,y) q(x,y)\, dy\, dx\, dt.$$

By this way, we see that the noise $\dot{W}^Q$ is uncorrelated in time whereas it is correlated in space through the kernel q which is assumed to be regular. In our simulations, the kernel is gaussian and given by

$$q(x,y) = \frac{1}{4\xi^2} e^{-\frac{\pi}{4\xi^2}|x-y|^2}.$$

In this expression, ξ corresponds to the correlation length (roughly speaking, if two points are closer than ξ, then there is correlation between the random fluctuations observed at these two points).

The motivation for introducing a stochastic term in the equation came from our concern to model cardiac pathologies. In the cardiac muscle, tachyarrhythmia is a disturbance of the heart rhythm in which the heart rate is abnormally increased. This is a major trouble of the cardiac rhythm since it may lead to rapid loss of consciousness and to death. As explained in [JC06], the vast majority of tachyarrhythmia are perpetuated by a reentrant mechanism. In our study, we show numerically that reentrant patterns such as spiral or meander may be generated and perpetuated only by the presence of noise.

In our simulations, we considered different ionic models: FitzHugh-Nagumo model, Mitchell-Schaeffer model and Barkley model [Bar91] which is given by

$$f(V_\mathrm{m}, w) = V_\mathrm{m}(1 - V_\mathrm{m})(V_\mathrm{m} - \frac{w+b}{a}),\ g(V_\mathrm{m}, w) = V_\mathrm{m} - w. \tag{3.24}$$

For Barkley model, similar simulations showing reentrant patterns have already been presented in [Sha05]. Compared to this study, one of the main differences of our work lies in the numerical discretization of our problem and especially of the Q-Wiener process: the discretization completely relies on finite elements. Thus, contrary to other numerical studies ([ANZ98], [Yan05]...), we do not consider a Galerkin spectral method or exponential integrator, that is we neither use the spectral decomposition of the solution of (3.23) in a Hilbert basis nor the semigroup attached to the linear operator (the Laplace operator in (3.23)), in order to build our scheme. This allows to apply our method on arbitrary geometries. This kind of discretization based on finite element methods was already considered in theoretical papers like [DP09], [KLL12] and [Kru14].

5.2 Methods and results

To discretize in space the problem, we denote by E^h the finite element space. The hypotheses made on the noise allow to get a version of $(W_t^Q(x),\ (t,x) \in \mathbb{R}^+ \times D)$ which is continuous in time and regular in space, thus the noise can be approximated thanks to the classical interpolation operator in the finite element space. For instance, for the FE method with Lagrange P^1 elements, we approximate W_t^Q for any t by

$$W_t^{Q,h} = \sum_{i=1}^{N_h} w_i \, \psi_i \tag{3.25}$$

where $(\psi_i)_{1 \leqslant N_h}$ is the P^1-FE basis and the family $(w_i)_{1 \leqslant i \leqslant N_h}$ is a centered Gaussian vector with covariance matrix $(tq(P_i, P_j))_{1 \leqslant i,j \leqslant N_h}$.

To discretize the problem in time, we define the time step $\Delta t = \frac{T}{N}$ for $N \geqslant 1$ and $t_n = n\Delta t$ for $0 \leqslant n \leqslant N$. The temporal scheme relies on the implicit Euler-Maruyama scheme (a generalization of the classical Euler scheme to the stochastic equation) with an explicit treatment of the reaction term.

The approximation at time t_{n+1} of (V_m, w) solution of (3.23) is given by $(V^h_{\mathrm{m},n+1}, w^h_{n+1})$ solution in $E_h \times E_h$ of:

$$\begin{cases} (V^h_{\mathrm{m},n+1} - V^h_{\mathrm{m},n}, \Psi^h) + \nu \Delta t (\nabla V^h_{\mathrm{m},n+1}, \nabla \Psi^h) = \dfrac{1}{\epsilon} \Delta t (f(V^h_{\mathrm{m},n}, w^h_n), \Psi^h) + \sigma(\Delta W^{Q,h}_{n+1}, \Psi^h), \, \forall \, \Psi^h \in E_h \\ w^h_{n+1} = w^h_n + \Delta t g(V^h_{\mathrm{m},n+1}, w^h_n) \end{cases} \tag{3.26}$$

with the Wiener increment given by $\Delta W^{Q,h}_{n+1} = W^{Q,h}_{n+1} - W^{Q,h}_n$. The law of the random variables $\Delta W^{Q,h}_{n+1}$ is given by

$$\Delta W^{Q,h}_{n+1} \sim \sqrt{\Delta t} (W^{Q,h}_1)_n$$

where $((W^{Q,h}_1)_n)_{1 \leqslant n \leqslant N+1}$ is a sequence of independent Q-Wiener process defined by (3.25) and evaluated at time 1.

If this scheme is applied to the semilinear heat equation, it has been proved recently in [Kru14] the following error estimate

$$\sqrt{\mathbb{E}(\|V^h_{\mathrm{m},n} - V_\mathrm{m}(t_n)\|^2_{L^2(D)})} \leqslant C(h + \sqrt{\Delta t}), \forall \, 0 \leqslant n \leqslant N.$$

Let us now describe the results that we have obtained with this discretization scheme. Figures 3.8 and 3.9 display the numerical simulations obtained with the model (3.23)-(3.24). In both cases, we do not have any deterministic stimulation which activates the front and we assume that noise acts on the whole domain.

In Figure 3.8, we have considered periodic boundary conditions. From some time, we observe the spontaneous generation of waves with a reentrant pattern. Some zone of the spatial domain emerges as a spontaneous pacemaker and generates a front which propagates on the whole domain. The reentrant wave is self-sustained: a previously activated zone is re-activated by the same wave. This front seems periodic and seems to organize itself in a tidy fashion despite the maintained action of the stochastic term.

In Figure 3.9, we have considered the same model complemented with homogeneous Neumann boundary conditions on a cardioid domain. We observe the spontaneous generation of a wave turning around itself like a spiral and thus reactivating zones already activated by the same wave.

In both simulations, we observe reentrant patterns such as spirals. As explained in [JC06], these phenomena are known to be source of tachycardia in the heart tissue.

In our work, we have also studied the influence of the noise intensity σ and of the timescale ϵ through a bifurcation diagram (Figure 3.10). We observe that, σ being fixed, if ϵ is small enough (that is, if the transition between the quiescent and excited state is enough sharp), we do not

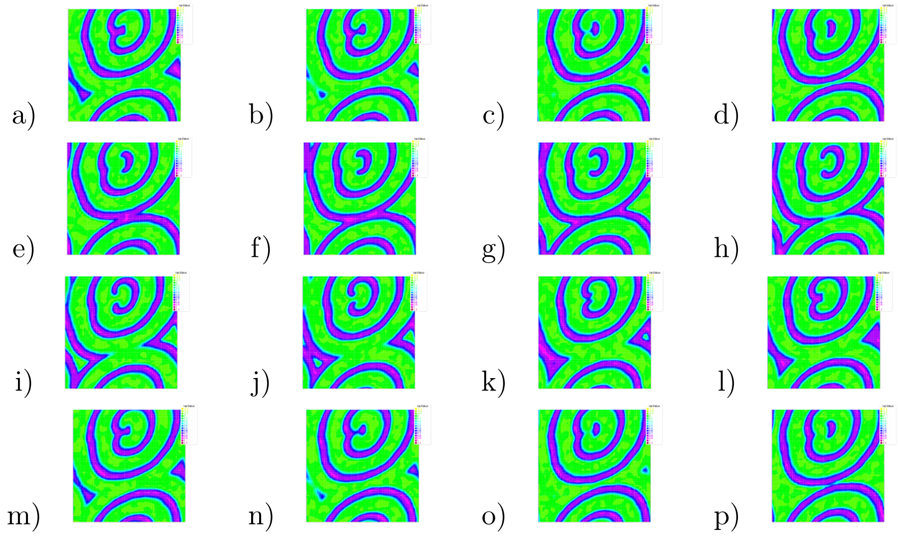


Figure 3.8: Simulations of system (3.23)-(3.24) with $\xi = 2$, $\sigma = 0.15$, $\varepsilon = 0.05$, $a = 0.75$, $b = 0.01$ and $\nu = 1$. These figures must be read from the top-left to the bottom-right. The quiescent state is represented in green whereas the excited state is in violet. There is $0.5ms$ between each snapshot.

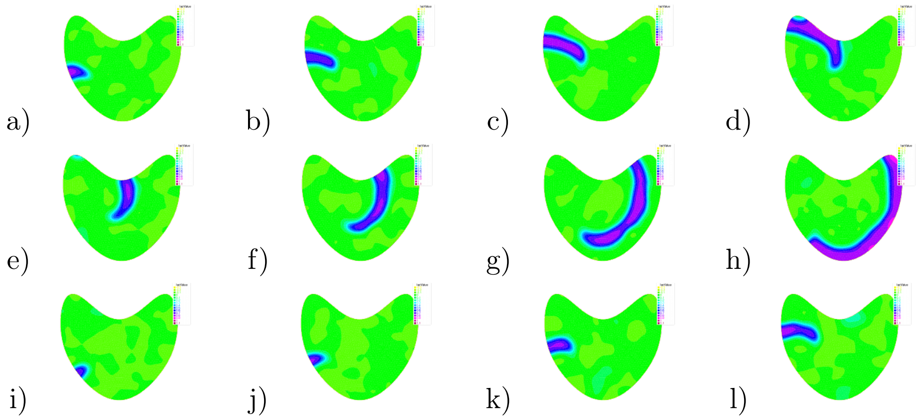


Figure 3.9: Simulations of system (3.23)-(3.24) with $\xi = 2$, $\sigma = 0.15$, $\varepsilon = 0.05$, $a = 0.75$, $b = 0.01$ and $\nu = 1$. The quiescent state is represented in green whereas the excited state is in violet. There is $2ms$ between each snapshot.

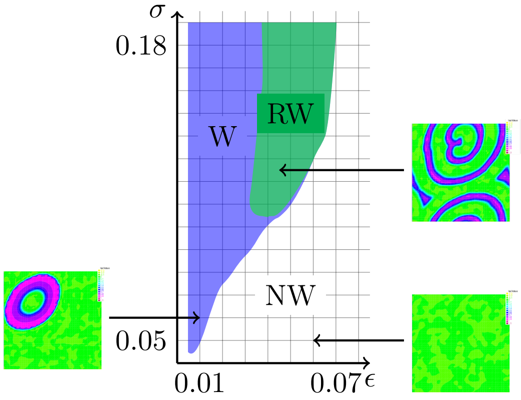


Figure 3.10: Numerical bifurcation diagram between ϵ and σ.

observe reentrant patterns anymore but only one wave which is not self-maintained.

We now end with the presentation of some perspectives for this work. In [TJ10], the authors present simulations with the Hodgkin-Huxley model in the case of a one dimensional nerve fiber. This fiber is stimulated with a signal which is decomposed as a deterministic input and a noise of small intensity. They show, through numerical simulations, that the presence of small noise may annihilate the generation of repetitive spiking if the intensity of the deterministic input is close to the threshold intensity to generate a spike. On the other hand, the stochastic resonance corresponds to the case when waves are sustained even for parameters values that do not support such behavior in the deterministic model. It could be interesting to observe these phenomena for our model. At last, our simulations could also be achieved with more realistic ionic models or with conductance-based noises.

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